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PROMETHEUS INC.

FINAL REPORT

POLYNOMIALS WITH RESTRICTED COEFFICIENTS AND THEIR APPLICATIONS

PREPARED BY:
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POLYNOMIALS WITH RESTRICTED COEFFICIENTS AND THEIR APPLICATIONS

PROMETHEUS INC.

ABSTRACT

Certain design restrictions growing out of antenna theory yield a beautiful class of complex variables problems. Our work was devoted to formulating these problems in mathematical terms, solving some of them, and beginning work on the others. One important result achieved was the development of a new method of estimating Gaussian-type exponential sums. Improvements of our previous results in null steering and notch filtering were also attained. Other findings were obtained in the areas of: the robustness of polynomials with unimodular coefficients; the effect of errors in such standard and crucial approximations as the far-field, Fresnel, and Doppler compensation; the effect of errors in the noise covariance matrix; and the Parabolic Equation Method in underwater acoustics.

STATEMENT OF WORK

1. Convert the Kahane "Gauss Sums + Probabilistic Choices" proof into a constructive one, and apply these polynomials to the design of low peak-factor signals and other engineering problems.

2. Investigate other possible methods of constructing Kahane-type polynomials.

3. Investigate employing the Byrnes and Kahane polynomials to design new reflection phase gratings, and consider the application of these ideas to the quieting of an objects response to radar and active sonar, and the design of baffles used to quiet machinery noise from submarines.

4. Consider the incorporation of the Prometheus methods of analytically choosing shading coefficients into adaptive antenna problems.

5. Determine the applicability of Kahane's methods to the Erdos problem involving polynomials with coefficients ± 1, and investigate the minimum L^∞ norm of such polynomials.

6. Investigate the minimization of the peak-factor in an inverse Fourier transform, given the amplitude spectrum.

7. Conduct research into whether polynomials based upon the Byrnes construction can be used to produce notched filters with more than one notch.

I. INTRODUCTION

Polynomials with restricted coefficients have been objects of intense interest throughout twentieth century mathematics and engineering. Despite the large amount of effort which has been directed towards analyzing their properties, many deep and fascinating questions remain. We continue our attack upon these problems, employing both traditional and novel methods. This report, with its appendices, presents the details of our recent work.

We have seen that certain design restrictions growing out of antenna theory yield a beautiful class of complex variables problems. Moreover these problems form an integral part of classical mathematical analysis, as the study of such polynomials was initiated by G. H. Hardy [46,p. 199], and furthered by J. E. Littlewood [28-30], P. Erdös [20], and many others. Our research continues to focus upon both aspects of this remarkable intertwining between the disciplines of pure mathematics and engineering.

The great success of J. P. Kahane [26] in solving the Littlewood conjecture, by showing that there indeed exist polynomials with unimodular coefficients whose modulus is essentially constant on the unit circle, lead us to reconsider the Erdös variant of this problem. Namely, do such polynomials exist when the coefficients are further restricted to be ± 1? Kahane's answer to Littlewood's question completed a series of researches by Beller [2-5], Byrnes [13, 14], Körner [27], Newman [3-5, 34], and Littlewood himself. As just mentioned the answer was an unqualified yes, near constancy for polynomials with unimodular coefficients is definitely available. In

contrast to this there has been very little progress in answering Erdös' question, and this has been one focus of our research.

We are conducting a two pronged attack upon the Erdös question. One tack we are taking is to attempt to exploit the breakthrough of Kahane, which we believe was due to his ingenious use of randomness and probability in his construction. Behind his and previous approaches was the idea of Gauss, viz. the "Gauss Sums." To put it quite simply we feel that Littlewood's problem was vanquished by the "equation"

Kahane=Gauss Sums + Probabilistic Choices

Since the Gauss Sums can also be written as polynomials with coefficients ± 1, we devoted considerable study to such sums. Our entirely new approach to estimating them, together with the results achieved thusfar, are described below.

The above describes, to some extent, the first prong of our attack on the Erdös problem. The second is our approach to the possibility of a negative answer to the Erdös question. Indeed numerical evidence seems to favor this choice. A computational scheme based upon the Bose-Einstein model for Statistical Mechanics has produced calculations of autocorrelations which seem to insist that the L^4 norm of a ± 1 type polynomial is $>(6/5)^{1/4}(n)^{1/2}$, and this if true would surely yield the "no" answer to the Erdös problem. Since this Bose-Einstein model can be followed up theoretically, and not only numerically, we envision a possible proof of the negative answer, if such is the case. What we have proven thusfar is that the L^4 norm of the well-known Shapiro polynomials of ± 1 type is $(4/3)^{1/4}(n)^{1/2}$

asymptotically. Details of these results appear below, and in Appendix B.

The locations of zeroes of polynomials with restricted coefficients is of interest from both a theoretical and a practical point of view. Under some circumstances, such as those encountered in peak power limited transmitting [40], the design of low peak-factor [39] and low crest factor [7] signals, and the design of digital test signals [36], one wants the modulus of the polynomial to be as close to constant as possible on the unit circle. It seems reasonable that such a condition would be achieved by polynomials whose zeroes are as far as possible from the unit circle. Certain applications, on the other hand, require zeroes at specific locations on the unit circle. For example, the classical mathematical problem in notch filter design is to produce a polynomial whose magnitude on the unit circle is close to constant in almost all directions, but which has a small number of deep nulls ("notches") at specified points. In [16] the construction of [13] is employed to produce such a polynomial having one null, with the added feature that all coefficients have the same magnitude. We have expanded upon these methods to produce multiple notches, while maintaining the near constancy of the magnitude of the polynomial at all points on the unit circle excluding small neighborhoods of the notches. Again, the results appear below.

Null steering is a crucial application which requires locating zeroes at specific points on the unit circle. There are several factors which must be considered in the design of null steering algorithms. In addition to the basic problem of placing the nulls the

main beam must be steered, the width of the main lobe controlled, and the sidelobe levels must be sufficiently below that of the main lobe. Control of the sidelobe level is usually achieved by attenuating the shading coefficients as one moves away from the center of the array. Often these attenuation factors (Chebyshev, Taylor, etc.) are chosen in advance, and may not be easily altered once the array is in place. This leads directly to a beautiful mathematical question, similar to the peak-factor problem in engineering discussed earlier, which we have attacked by the methods employed to study polynomials with coefficients of magnitude one:

> Given the magnitude of the coefficients of a polynomial P, a finite subset S of the unit circle C, and a point $p \in C$ distinct from those in S, choose the phases of these coefficients so that $P(z)=0$ for all $z \in S$, the maximum on C of $|P(z)|$ occurs at $z=p$, and the maximum of $|P(z)|$ on a subset of C excluding an appropriate interval (the beamwidth) around p is as small as possible.

Currently the most widely used class of null steering methods is known as adaptive nulling [1,9,24,25,32,45]. Adaptive arrays have developed over the past twenty-five years as the preferred method of reducing the performance deterioration in signal reception systems which is inevitably caused by undesired noise entering the system. Sources for this noise include multipath affects, electronic countermeasures, clutter scatterer returns, antenna location errors, array element thermal noise, etc. The proliferation of such noise sources has greatly increased the importance of interference

suppression in essentially all applications. Although such adaptive methods as the Widrow least mean squares (LMS) and Howells-Applebaum sidelobe canceller have achieved considerable success, difficult problems remain. Foremost among these are poor transient response, signal cancellation resulting from interaction between signal and interference, excessive computation time, and sidelobe degradation when jammer cancellation is attempted. A secondary problem is the lack of control in adaptive algorithms of the dynamic range of the weights.

These methods are indirect adaptive schemes, they do not explicitly form an estimate of the directions of arrival of interfering sources or explicitly steer nulls in those directions. A scheme in which these two tasks are actually performed can be called a direct adaptive algorithm. Thus one approach to the solution of such problems is to complement an appropriate indirect adaptive algorithm with the analytic null steering methods described in [17]. In this way the actual noise suppression achieved can be enhanced beyond that which would be available through either adaptive or analytic methods exclusively. We have begun to explore the poesibility of this "Direct Adaptive Antenna System," and will continue this work during Phase II.

Other questions which have been the focus of our research effort, and which we report on below, include: the robustness of polynomials with unimodular coefficients; the effect of errors in such standard and crucial approximations as the far-field, Fresnel, and Doppler compensation; the effect of errors in the noise covariance matrix; and the Parabolic Equation Method in underwater acoustics.

II. WEYL SUMS

In connection with the beamforming problems under consideration, we were required to obtain certain trigonometric sum estimates. In particular, sums such as

$$\sum_{k\in I} e^{it k^2}$$

made their appearance, and we desired "Gaussian-like" estimates for these "Gaussian-like" sums. In short, we wanted the bound of $O\sqrt{n}$ for said sums taken over intervals, I, of length of order n and for t of the order of 1/n.

The available method for handling such sums is that of Hermann Weyl. Realizing the intractability of sums of exponentials of quadratics, he reduced such sums to exponentials of linears (geometric sums!) by simply multiplying by their conjugates. Unfortunately, however, his method leads to bounds like $O\sqrt{n \log n}$ instead of the $O\sqrt{n}$ that we desire.

To see the emergence of this logn factor, let us examine his procedure in some detail. Assume that I is an interval of length $\leq n$ and write

$$S = \sum_{k\in I} e^{it k^2} \text{ so that}$$

$$|S|^2 = S\bar{S} = \sum_{k\in I} e^{it k^2} \sum_{j\in I}' e^{-it j^2} = \sum_{k,j\in I} e^{it(k^2 - j^2)}.$$

If we then write k-j=v, so that k+j=2k-v, and call I_v the intersection of the two intervals, I and I translated by +v, we obtain

$$(1.)\quad |S|^2 = \sum_{v=-(n-1)}^{n-1} e^{-i\pi v^2} \sum_{k\in I_v} e^{2i\pi vk}.$$

But here is the rub: one has the choice of throwing away the factor $e^{-i\pi v^2}$ by "crashing through" and thereby losing accuracy, or being stuck with the same intractable sum of exponentials of quadratics. Weyl opted for the "crashing through." The resulting bound is (with $n_v=|I_v|$),

$$\sum_{v=-(n-1)}^{n-1} \left|\frac{\sin n_v t v}{\sin tv}\right| \le n+2\sum_{v=1}^{n-1}\frac{1}{|\sin tv|} \le n+\frac{\pi}{t}\sum_{v=1}^{n-1}\frac{1}{v} \quad (\text{if } 0<t<\frac{\pi}{2n}, \text{ e.g.})$$

and thereby emerges the onerous logn.

Our approach, then, is to make the other choice and face the troublesome $e^{-i\pi v^2}$, the trick being that this sum, though still intractable, can be directly related to our original sum. An estimate of the desired quantity is given <u>in terms of itself</u>, and this will prove effective.

So let M be the exact bound for $\sum_{k\in I} e^{iP(k)}$

over all intervals, I, of length $\le n$ and all real quadratic polynomials, P(k), with leading coefficient $t\neq 0$. We will first treat the restricted case where $0<t\le \pi/2n$. The general case will follow from this one since this says $n\le \pi/2t$ so that a general sum obviously

splits into $\lceil 2nt/\pi \rceil$ of these restricted ones. Our aim is to show here that $M=O(t^{-1/2})$ and this is the promised "Gauss-like" bound since our real interest is in the case of t of exact order 1/n.

To begin with we need the following well known facts.

Lemma 1 (Summation by parts):

$\sum_{k=1}^{n} a_k b_k \ll PS(a_k)\ V(b_k)$ where PS (partial sum bound) is

$\max_{m\le n} \left| \sum_{k=m} a_k \right|$, and V (total variation) is

$$\min(|b_1|, |b_n|) + \sum_{k=1}^{n-1} |b_{k+1}-b_k|.$$

(In particular, if the b_k are monotonic and of the same sign, then $V(b_k) = \max(|b_1|, |b_n|)$).

Lemma 2 (Gibbs' phenomenon) The partial sums of the series

$$\sum_{k=1}^{\infty} \frac{\sin k\theta}{k}$$

are uniformly bounded by 2 (not by $\pi/2$, which might be guessed since this is the Fourier series of $(\pi-\theta)/2$ on $0\le\theta\le 2\pi$).

Letting $I=[r+1, r+m]$, where $m\le n$, the same algebraic manipulations as before lead to

$$\left| \sum_{k\in I} e^{i(tk^2+Ak+B)} \right|^2 = m^2 + 2\,\mathrm{Im} \sum_{v=1}^{m-1} \frac{e^{-itv^2-iBv} - e^{itv^2-iAv}}{\sin vt}$$

with B = a+(2r+1)t, A = a+(2r+2m+1)t.

To estimate the right hand sum, we split it into

$$S_1 = \sum_{v<t^{-\frac{1}{2}}} \quad \text{and} \quad S_2 = \sum_{v\geq t^{-\frac{1}{2}}} .$$

For S_2, we use our Lemma 1, with the a_v the numerators, and the $b_v = 1/\sin vt$. The PS term is clearly bounded by 2M and the V term, by monotonicity, is bounded by

$$\frac{1}{\sin t^{\frac{1}{2}}} \leq \frac{\pi}{2t^{\frac{1}{2}}} .$$

Altogether, then, we get

$$S_2 << \pi M t^{-1/2}$$

To bound S_1, a different tack must be taken. We break S_1 further into the two sums

$$\mathrm{Im}\sum_{v<t^{-\frac{1}{2}}} \frac{e^{-itv^2-iBv}}{\sin vt} \quad \text{and} \quad \mathrm{Im}\sum_{v<t^{-\frac{1}{2}}} \frac{e^{itv^2-iAv}}{\sin vt} .$$

In both cases, the dropping of the quadratic term in the exponent causes an error or $<<v^2t$ in each term of the numerator. Thus, each of these sums is replaceable by a sum

$$T = \sum_{v<t^{-\frac{1}{2}}} \frac{\sin vC}{\sin vt} \quad \text{with an error}$$

$$<<\sum_{v<t^{-\frac{1}{2}}} \frac{v^2t}{\sin vt} \leq \frac{\pi}{2}\sum_{v<t^{-\frac{1}{2}}} \frac{v^2t}{vt} = \frac{\pi}{2}\sum_{v<t^{-\frac{1}{2}}} v < \frac{\pi}{2t} .$$

Finally, we must effect the estimation of our two sums T. This is achieved by writing it as

$$T = \frac{1}{t}\sum_{v<t^{-\frac{1}{2}}} \frac{\sin vC}{v} + \sum_{v<t^{-\frac{1}{2}}} \sin vC \left(\frac{1}{\sin vt} - \frac{1}{vt}\right).$$

The first of these is bounded by 2/t by Lemma 2, and crude methods suffice for the second sum. Since

$$\frac{1}{\sin x} - \frac{1}{x} < x \quad \left(\text{or } \sin x > \frac{x}{1+x^2}\right)$$

in the interval $(0, \pi/2)$ (fun for the reader?), we obtain, by crashing through, the estimate

$$\sum_{v<t^{-\frac{1}{2}}} vt \leq 1 \text{ for this second sum.}$$

Altogether, then, these many bounds combine to give

$$\left|\sum_{k\in I} e^{i(ck^2+ak+b)}\right|^2 \leq n + \pi M t^{-\frac{1}{2}} + \frac{4}{t} + \frac{\pi}{t} + 2,$$

and by picking a and m judiciously, this means that

$$M^2 \leq n + \frac{9}{t} + \frac{\pi}{\sqrt{t}} M.$$

Hence, as promised, we have estimated our bound, M, in terms of itself. Finally, then, we write this as

$$(M-\frac{\pi}{2\sqrt{t}})^2 \leq n+\frac{12}{t} \text{ and obtain } M \leq \frac{\pi}{2\sqrt{t}} + \sqrt{n+\frac{12}{t}} \leq \frac{6}{\sqrt{t}}$$

which is what we desired for the restricted sum. For the general sum we need only multiply by $\left\lceil \frac{2nt}{\pi} \right\rceil$ and obtain

$$M \leq \frac{6}{\sqrt{t}} + 4n\sqrt{t} \ .$$

Indeed, we have saved our precious logarithm!

Extension of the above results to the case where P(k) is a third degree polynomial, and then to arbitrary polynomials, will be pursued during Phase II.

III. L^4 NORMS

Investigations were continued into the L^4 norm of polynomials with coefficients ± 1. Revisions in the previous paper [35] were made, and the revised paper, which is Appendix B in this report, will appear in the American Mathematical Monthly.

To describe the new results, we again employ the notation of Appendix B. Thus,

$$P(z)=\sum_{k=0}^{n-1}\epsilon_k z^k, \text{ each } \epsilon_k=1 \text{ or } -1, \ z=e^{2\pi i\theta}, \ 0\le\theta<1.$$

As shown previously,

$$\|P\|_{L^4}^4=\sum_{\substack{j+k=\ell+m \\ 0\le j,k,\ell,m<n}}\epsilon_j\epsilon_k\epsilon_\ell\epsilon_m.$$

A straightforward computation yields the lower bound $n^2+2\lfloor n/2\rfloor$ for this sum. Also, by considering $-P(z)$ and $P(-z)$, it is clear that, without loss of generality, it may be assumed that $\epsilon_0=\epsilon_1=+1$. Using this, an exhaustive computer search for the minimum L^4 norm was made for $n\le 22$, on an IBM-XT compatible, with the results appearing in Table 1. Note that the lower bound $n^2+2\lfloor n/2\rfloor$ is achieved, with a unique combination of ϵ's, for $n=3,5,7,11,13$, and it is also achieved, but not uniquely, for $n=4$. Also observe that the number of computations required to evaluate the sum grows very rapidly with n, and the exhaustive search becomes impossible, even for the largest current computers, when n reaches about 30. Thus, other methods are necessary to decide the minimum L^4 norm of such polynomials and to possibly answer the Erdös question mentioned in [35]. Work on this problem is continuing.

# of epsilons: 3
optimum: 11

combinations:
+ + -

# of epsilons: 4
optimum: 20

combinations:
+ + + -
+ + - +

# of epsilons: 5
optimum: 29

combinations:
+ + + - +

# of epsilons: 6
optimum: 50

combinations:
+ + - - + -
+ + + - + -
+ + - + - -
+ + + + - +
+ + + - - +
+ + + - + +
+ + - + + +

# of epsilons: 7
optimum: 55

combinations:
+ + + - - + -

# of epsilons: 8
optimum: 80

combinations:
+ + + + - - + -
+ + - - - - + -
+ + + - - + - +
+ + + - + - - +

# of epsilons: 9
optimum: 105

combinations:
+ + + - + - + + -
+ + - - - - - + -
+ + + - - + + - +
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+ + + - + - - + +
+ + - - + - + + +

# of epsilons: 10
optimum: 126

combinations:
+ + - - - + - - + -
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# of epsilons: 11
optimum: 131

combinations:
+ + + - - - + - - + -

# of epsilons: 12
optimum: 164

combinations:
+ + + + + - - + + - + -
+ + - - + + + + + - + -
+ + + + - - + + - + - +
+ + + + - + - + + - - +

# of epsilons: 13
optimum: 181

combinations:
+ + + + + - - + + + - + - +

# of epsilons: 14
optimum: 234

combinations:
+ + + + - + - + + - - + + -
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# of epsilons: 15
optimum: 255

combinations:

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+ + + - - - + - - - + - - + -
+ + + + + + - - + + - - + - + -
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# of epsilons: 16
optimum: 304

combinations:

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+ + - + - - + - + + + - + + + -
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# of epsilons: 17
optimum: 353

combinations:

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+ + + - - - - - - - + - + - - + + -
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# of epsilons: 18
optimum: 374

combinations:

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+ + - - + + - - - - - - + - + + - + -
+ + + + - - - - - + - + - - + + - - +
+ + + + + + - + - - + - + + + - - + +
+ + - - + + + - + - - + - + - + - - +
```

# of epsilons: 19
optimum: 419

combinations:

```
+ + - + + - - + + + + - + - + - - - -
+ + + + - + - + - - - - - + + - - + - -
```

# of epsilons: 20
optimum: 452

combinations:

```
+ + + + + - + - - - + - + + - - - + + -
+ + - - + - - + + + + - + + + + - + - +
```

# of epsilons: 21
optimum: 493

combinations:

```
+ + - - - - - - - - + + - - + - + - + - - - +
```

# of epsilons: 22
optimum: 562

combinations:

```
+ + + + + + - - - + + - + - + - + + - - + -
+ + - - - + + - - + - - + - + - + - - - - -
+ + + + + - + - + - + + - + + - - + + + - -
+ + + - - + + + + + + - - + - - + - + - + +
+ + + + + - + + - - + - + - + + - - + + + +
+ + + - - - + - + - + - - + + - + + + + +
```

Table 1. $\|P\|_{L^4}^4$, 2 ≤ degree P ≤ 21

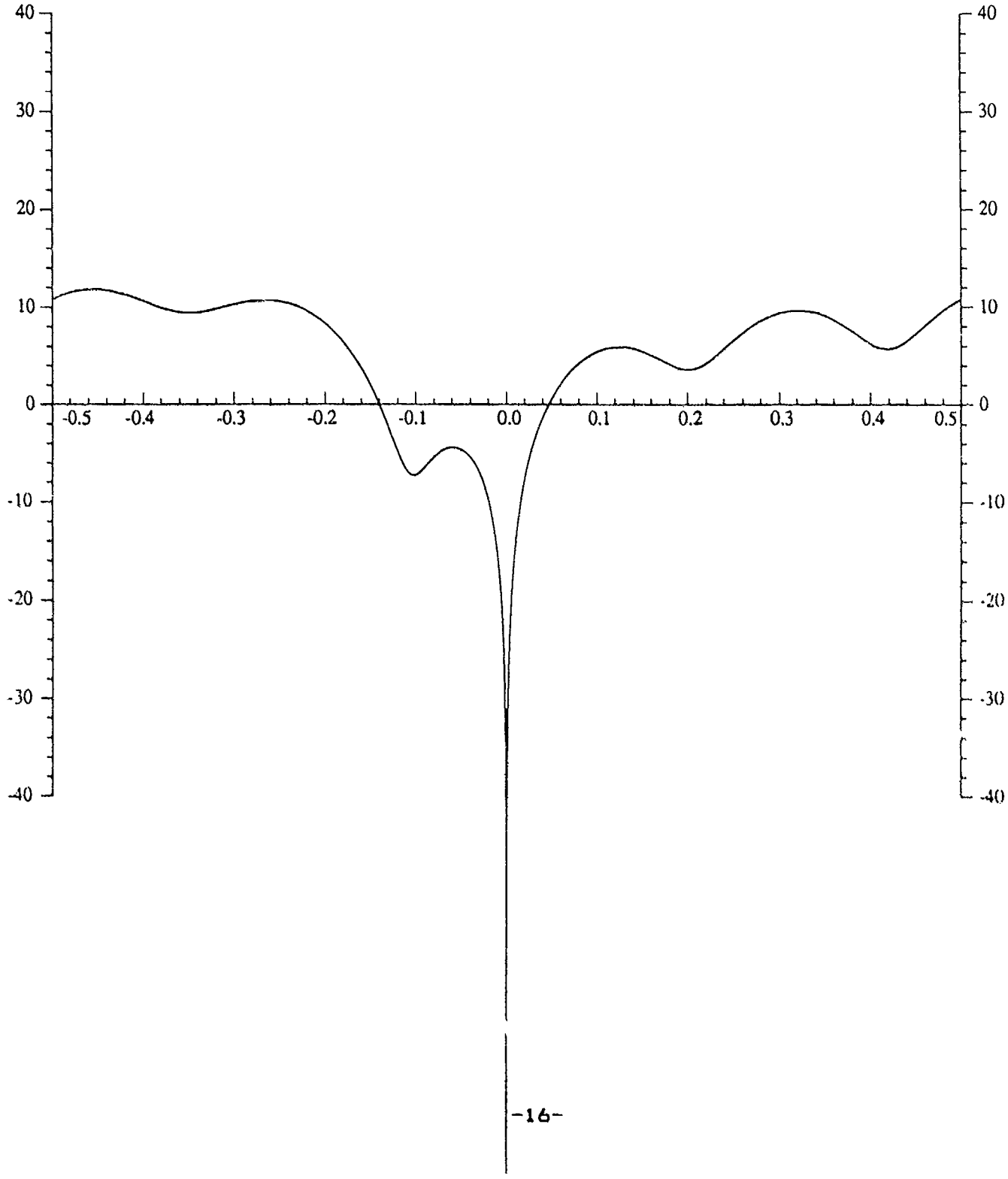
IV. NOTCH FILTERS AND NULL STEERING

Investigations were also continued into notch filters employing coefficients of equal magnitude. Revisions in the previous paper [16] were made, and the revised paper, which is Appendix D in this report, has been tentatively accepted (pending approval of the changes) for publication in the IEEE Transactions on Acoustics, Speech and Signal Processing.

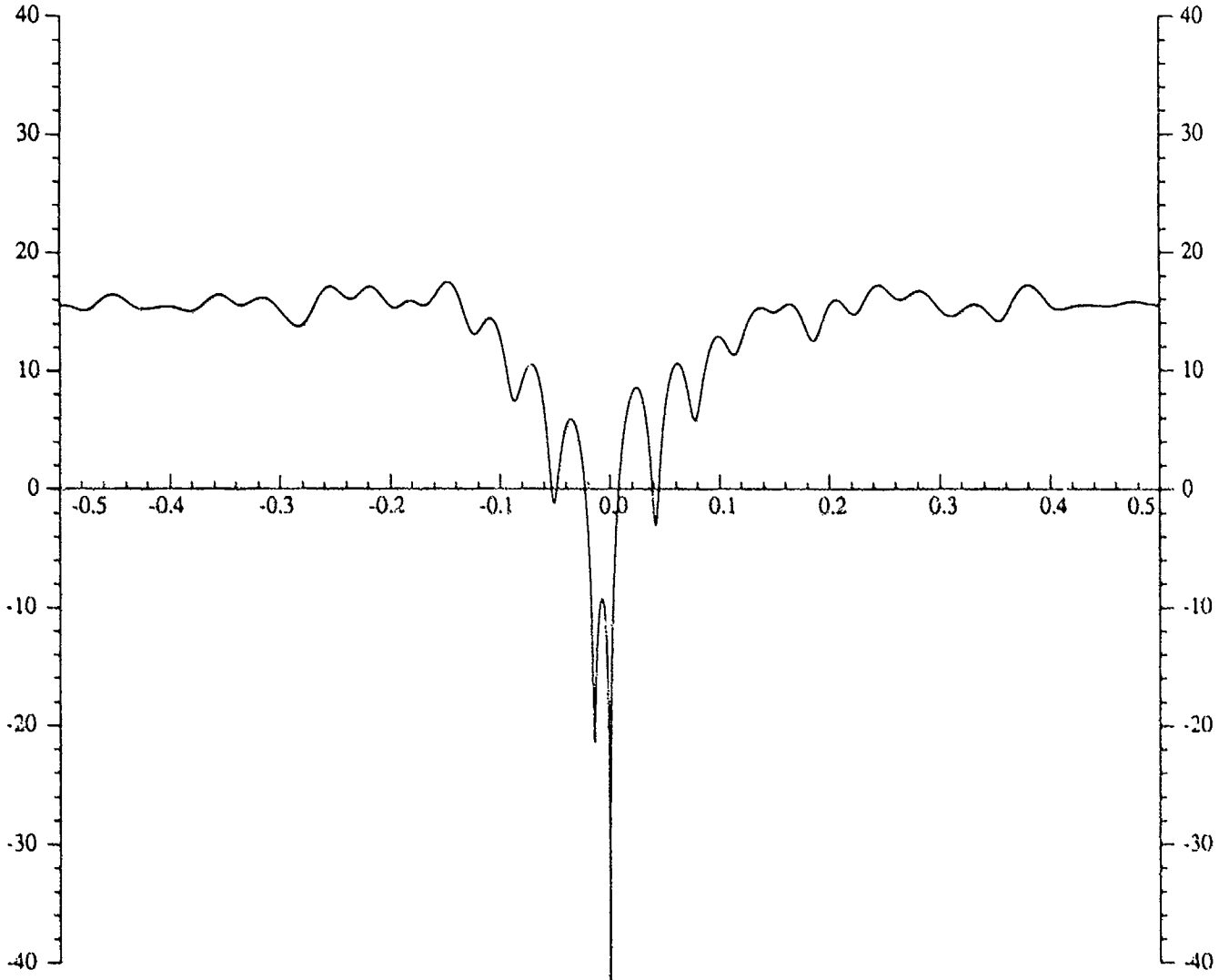
The new results in this area involve the design of notch filters with multiple notches, but still with dynamic range equal to 1. The basic idea is to again begin with the function $P(\theta)$ of [1], but instead of removing the first N terms (i.e., all terms with $k=0$), as was done in [1] to obtain $Q(\theta)$, all terms for several values of k are removed. This results each time in a notch filter, with the number of notches equal to the number of k's that are removed. Also, as all nonzero coefficients have magnitude one, these remain filters with unit dynamic range. The worth of this method may be seen in the graphs on the next 18 pages, which exhibit filters of degree 8 to 2024, with from 1 to 4 notches.

Another previous paper [17], dealing with the important subject of null steering, required revision. This was accomplished as part of the current effort, and the resulting paper, which is Appendix E in this report, will appear in the February 1988 issue of the IEEE Transactions on Antennas and Propagation.

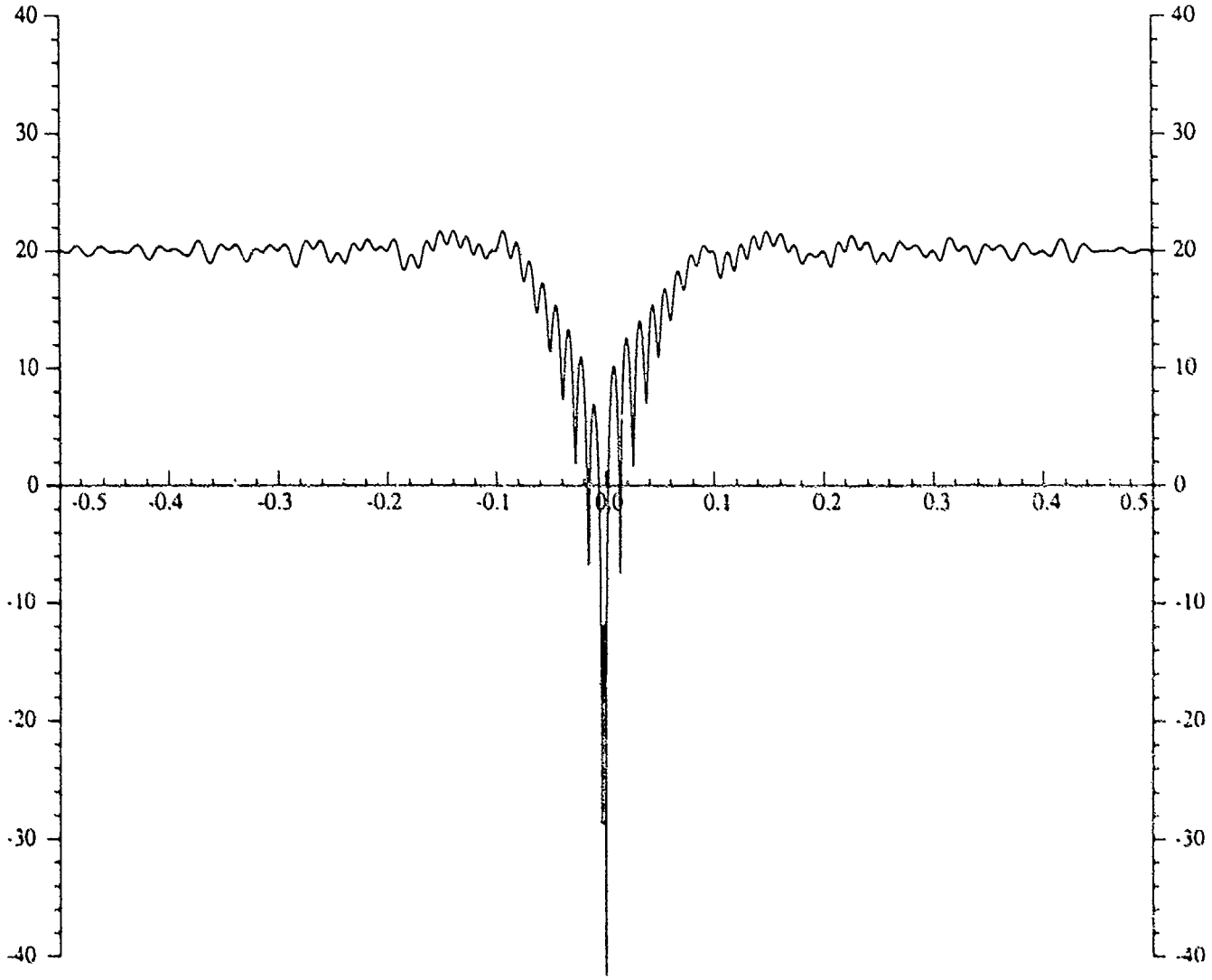
A Single Notch Filter, Degree 8



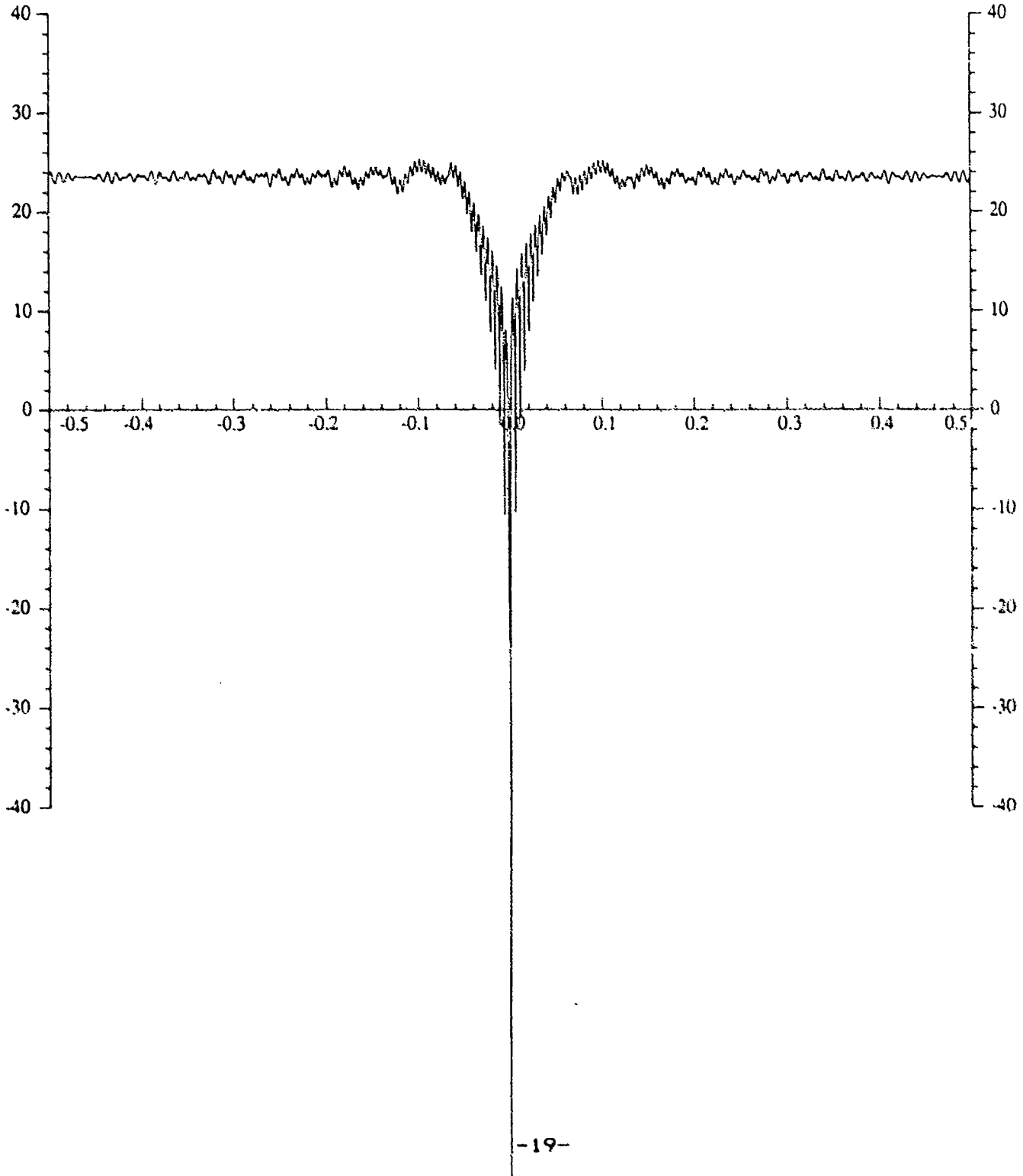
A Single Notch Filter, Degree 35



A Single Notch Filter, Degree 99

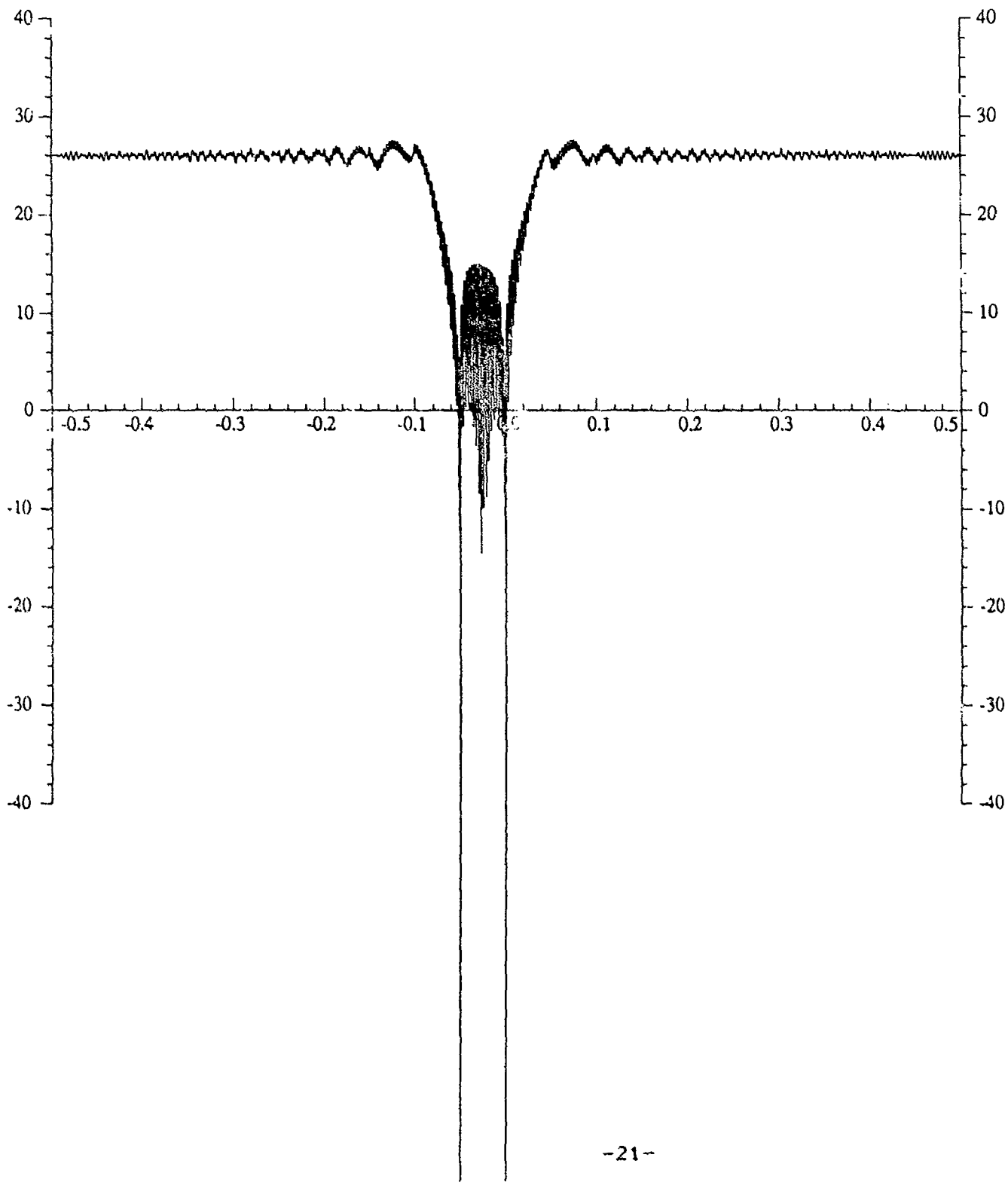


A Single Notch Filter, Degree 224

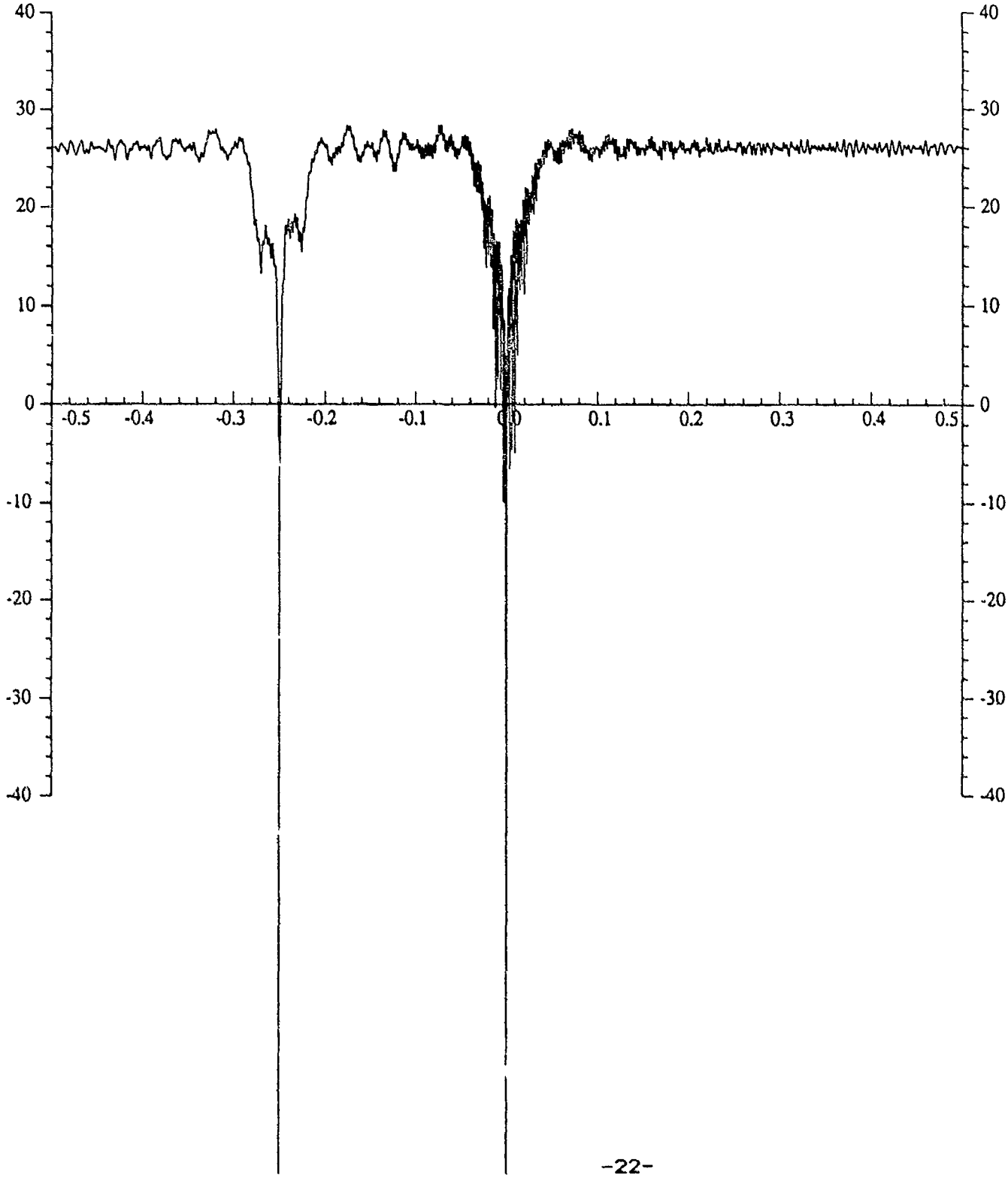


$|P(e^{i\theta})|$, Degree 483

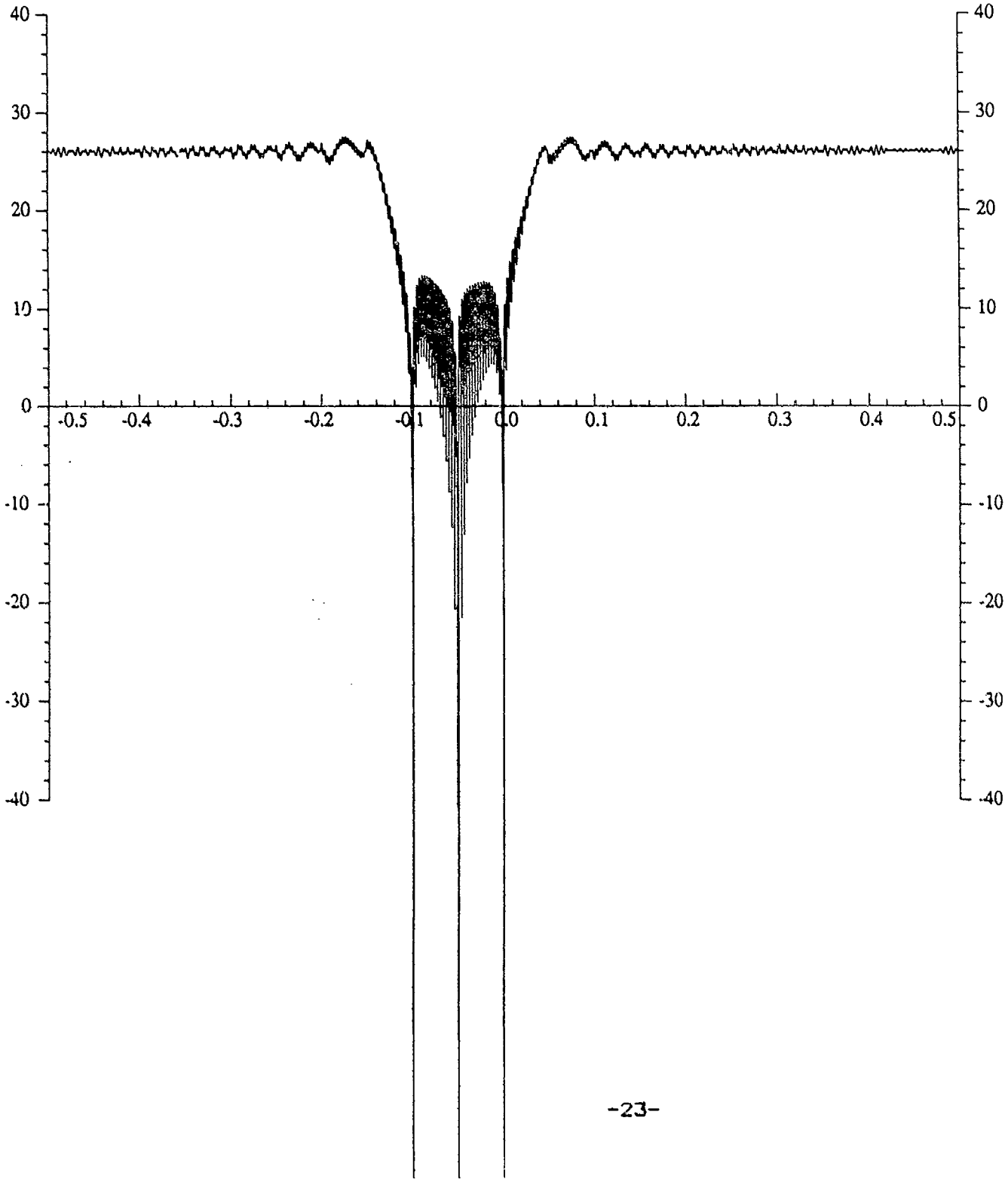
A Double Notch Filter, Degree 483



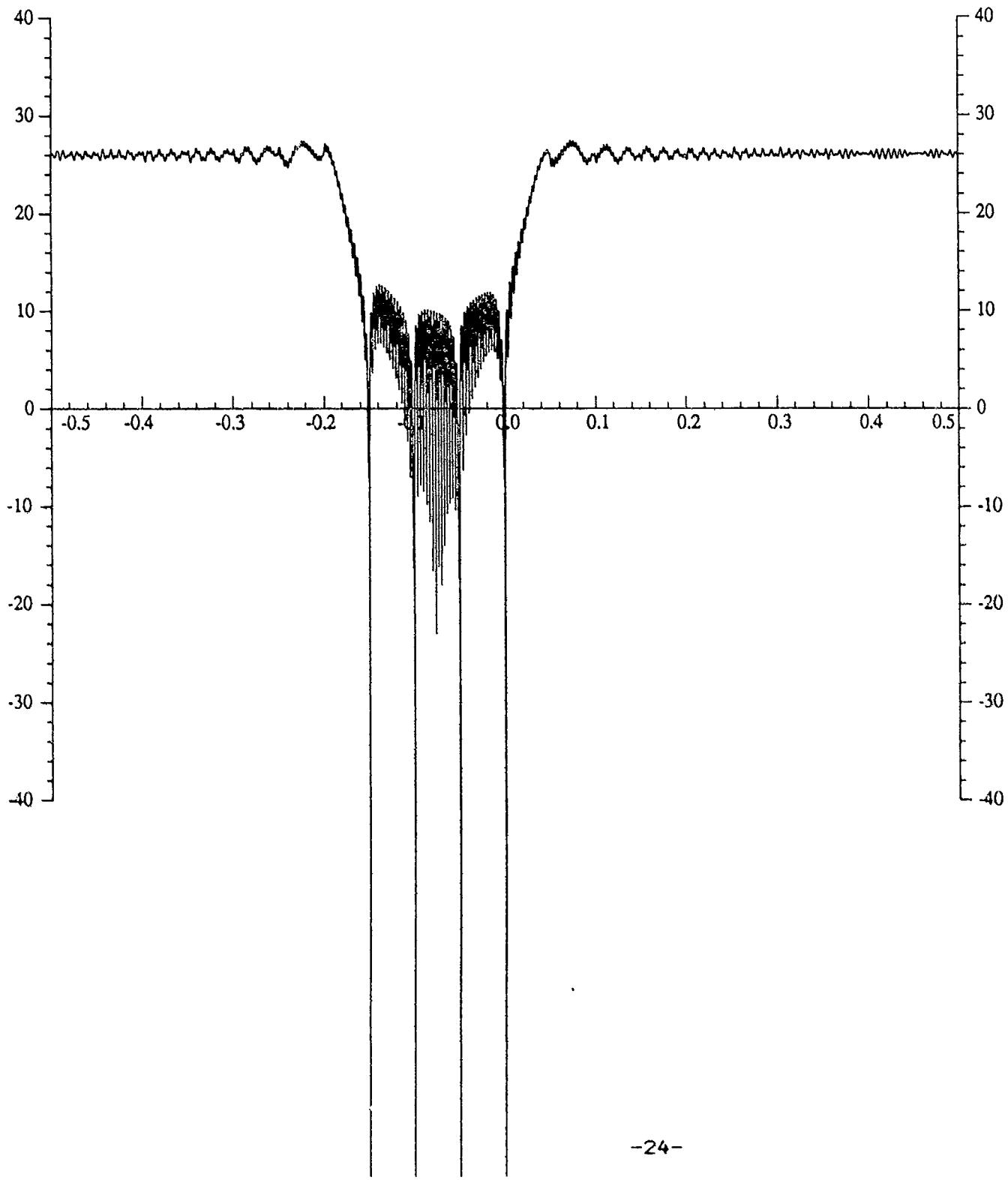
A Double Notch Filter, Degree 483



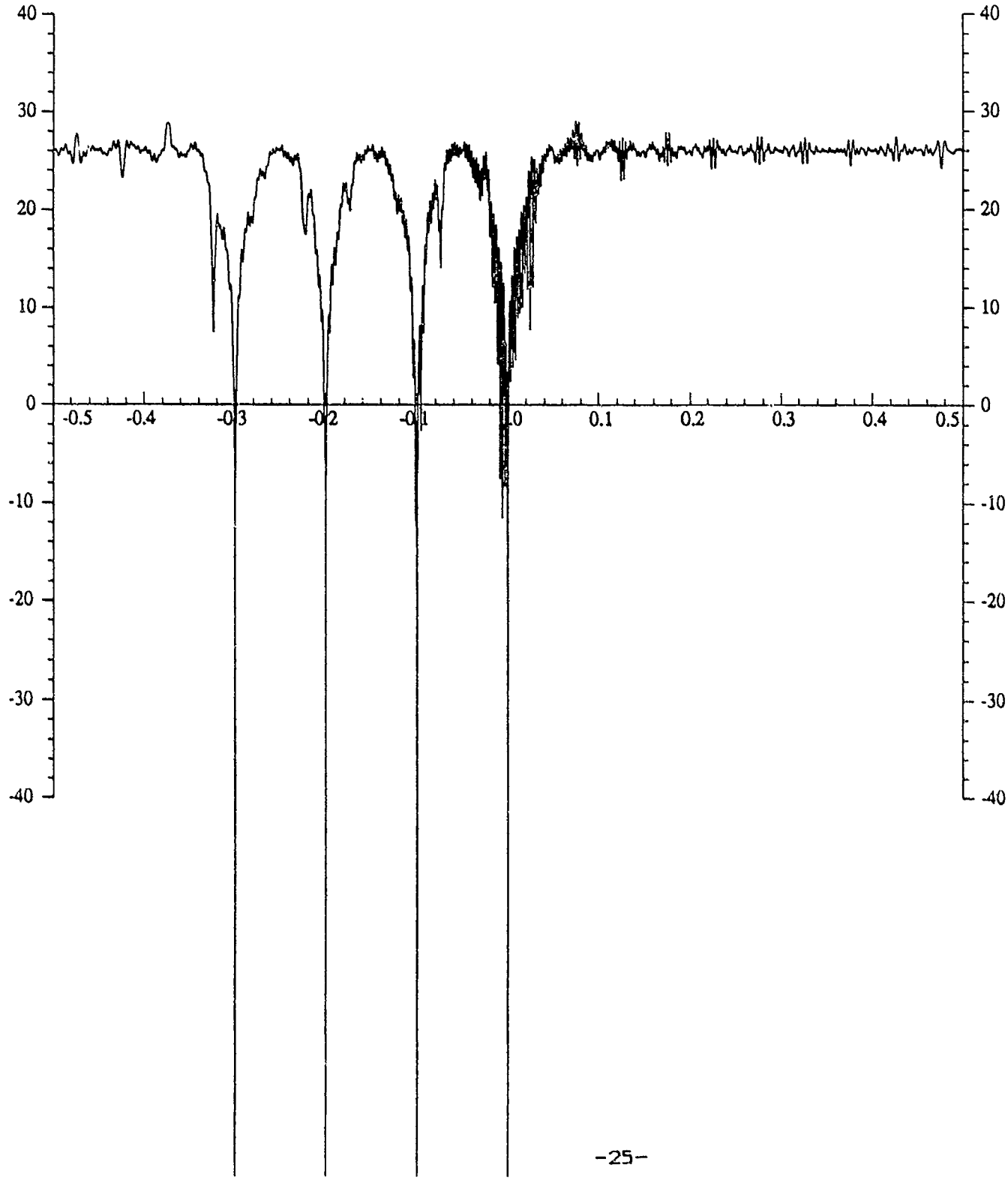
A Triple Notch Filter, Degree 483



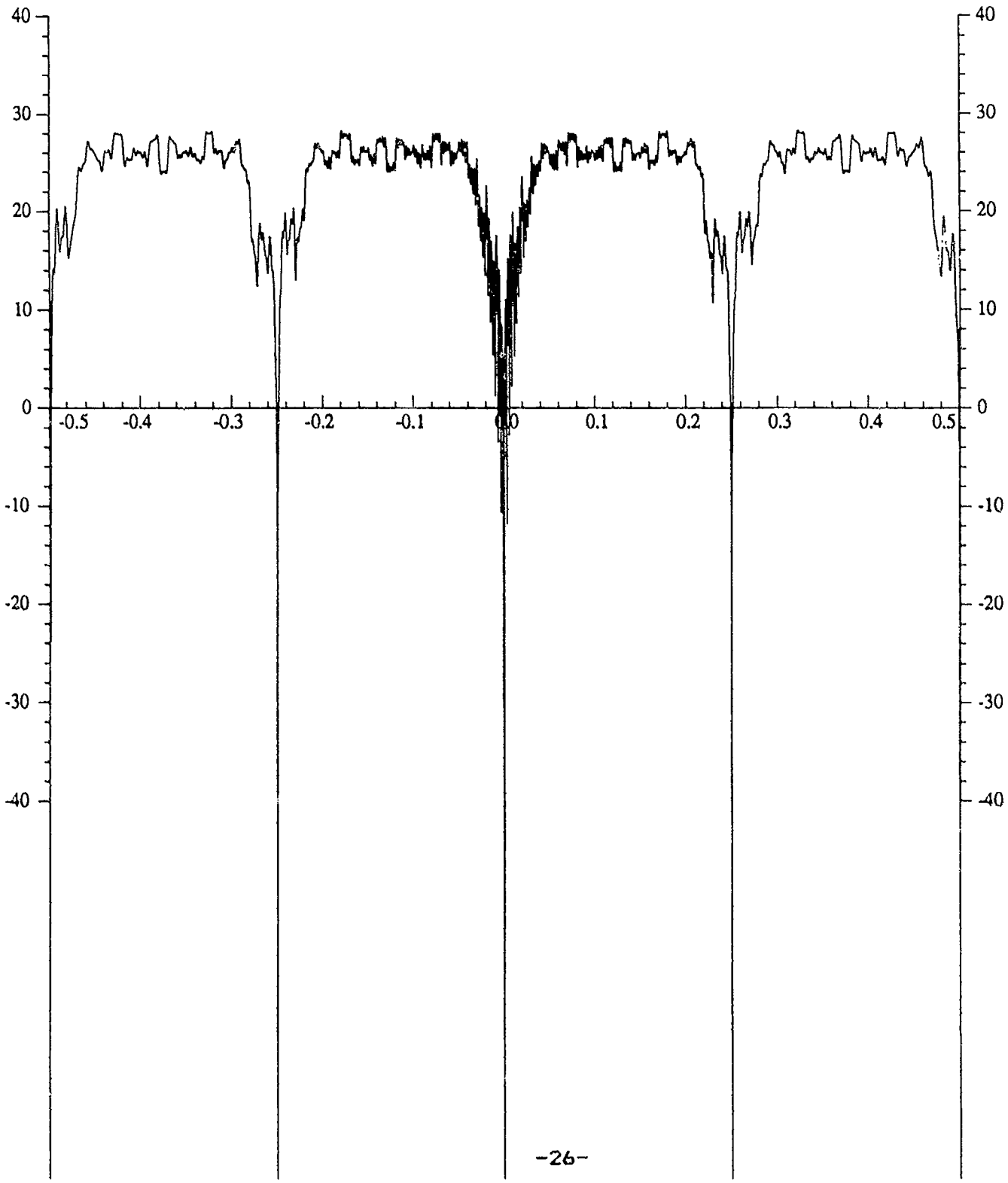
A Quadruple Notch Filter, Degree 483



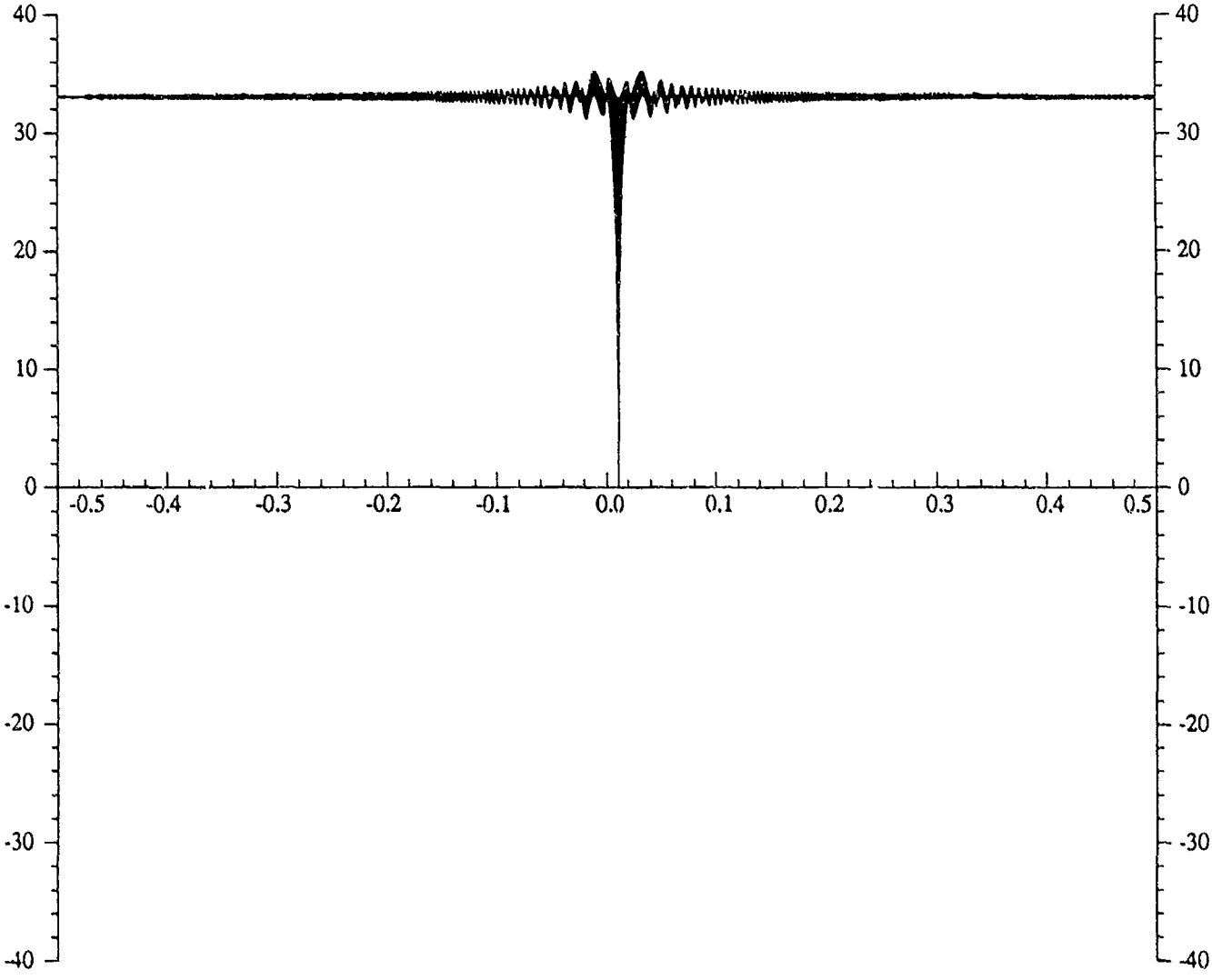
A Quadruple Notch Filter, Degree 483



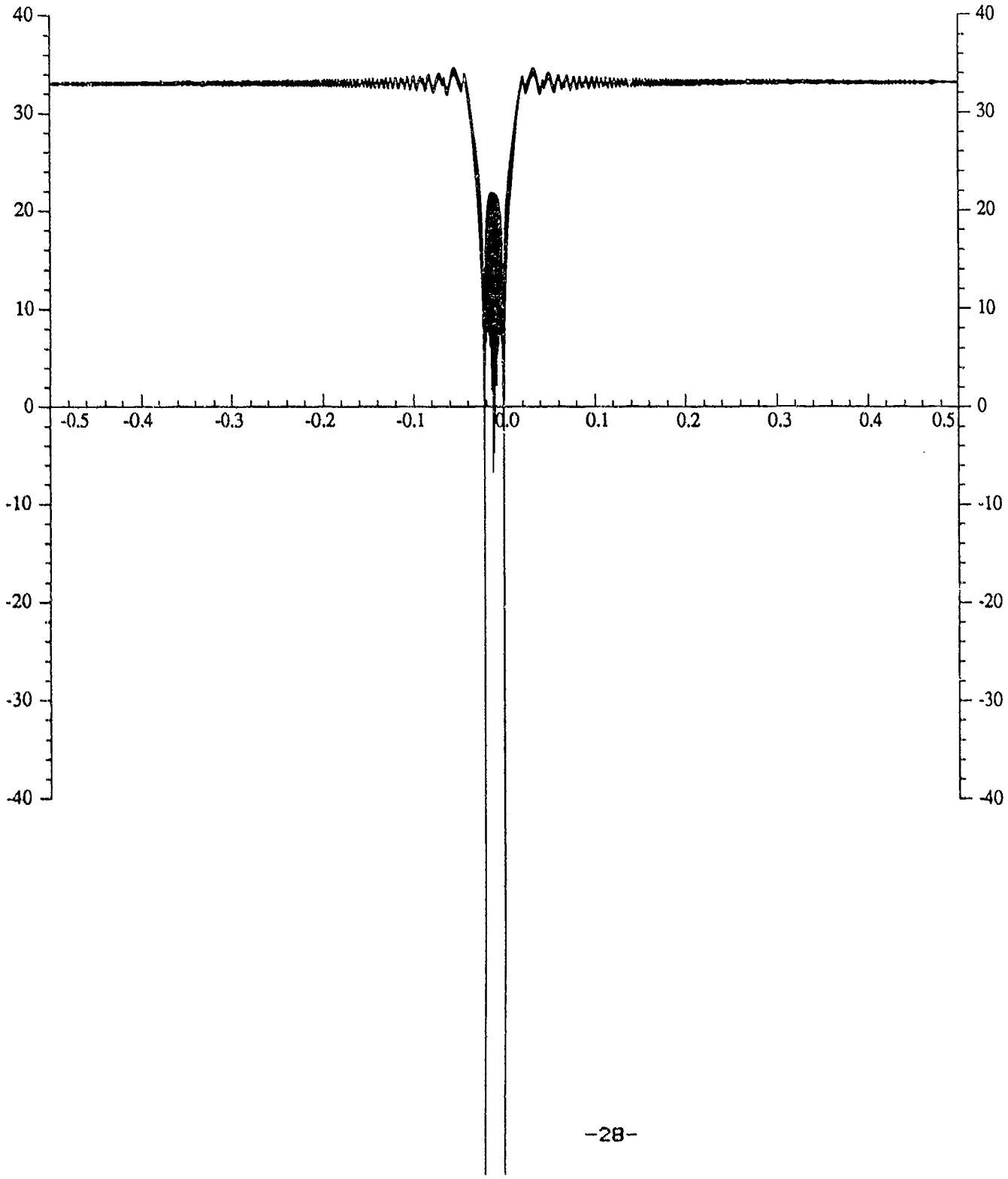
A Quadruple Notch Filter, Degree 483



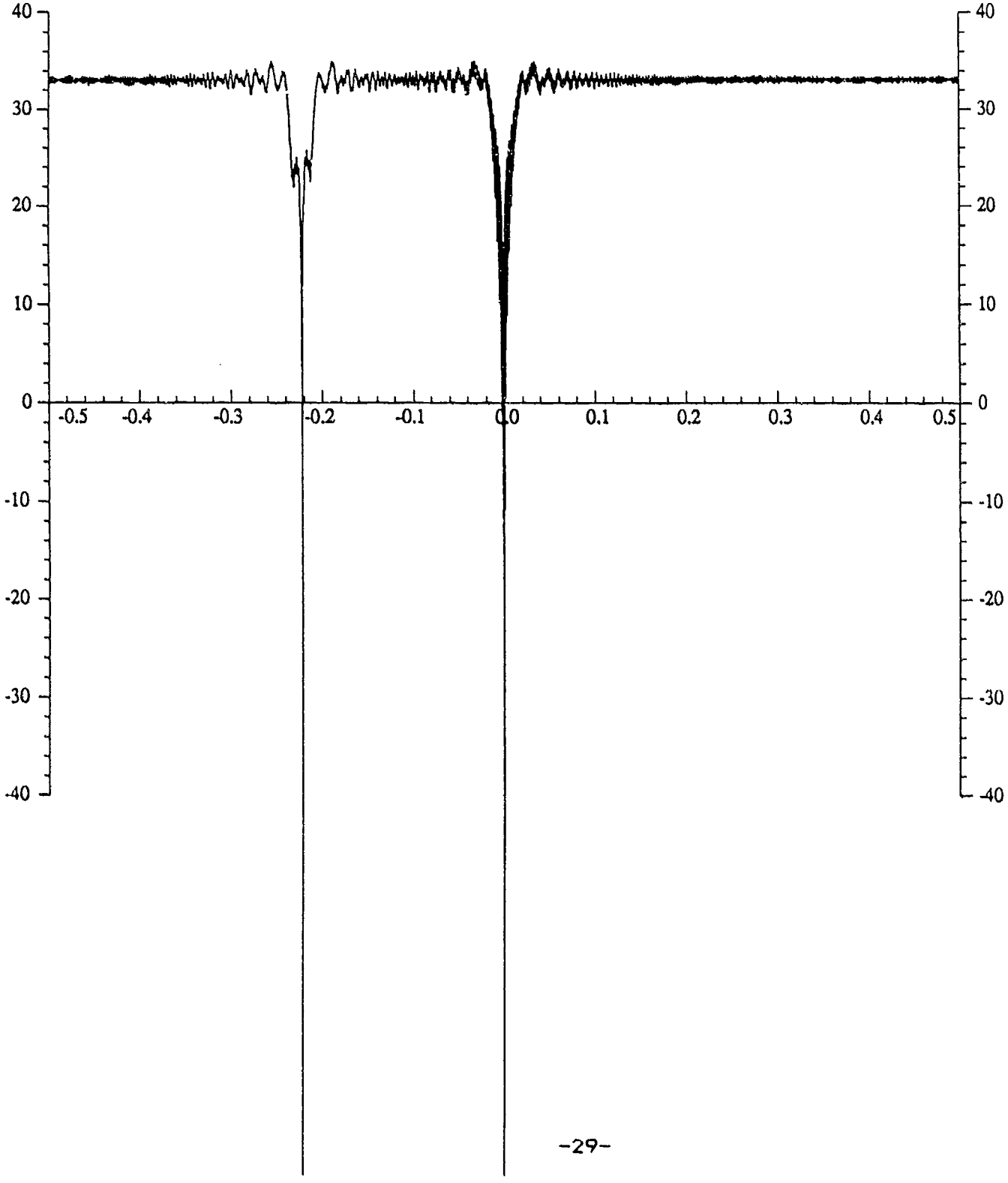
|P(θ)| , Degree 2024



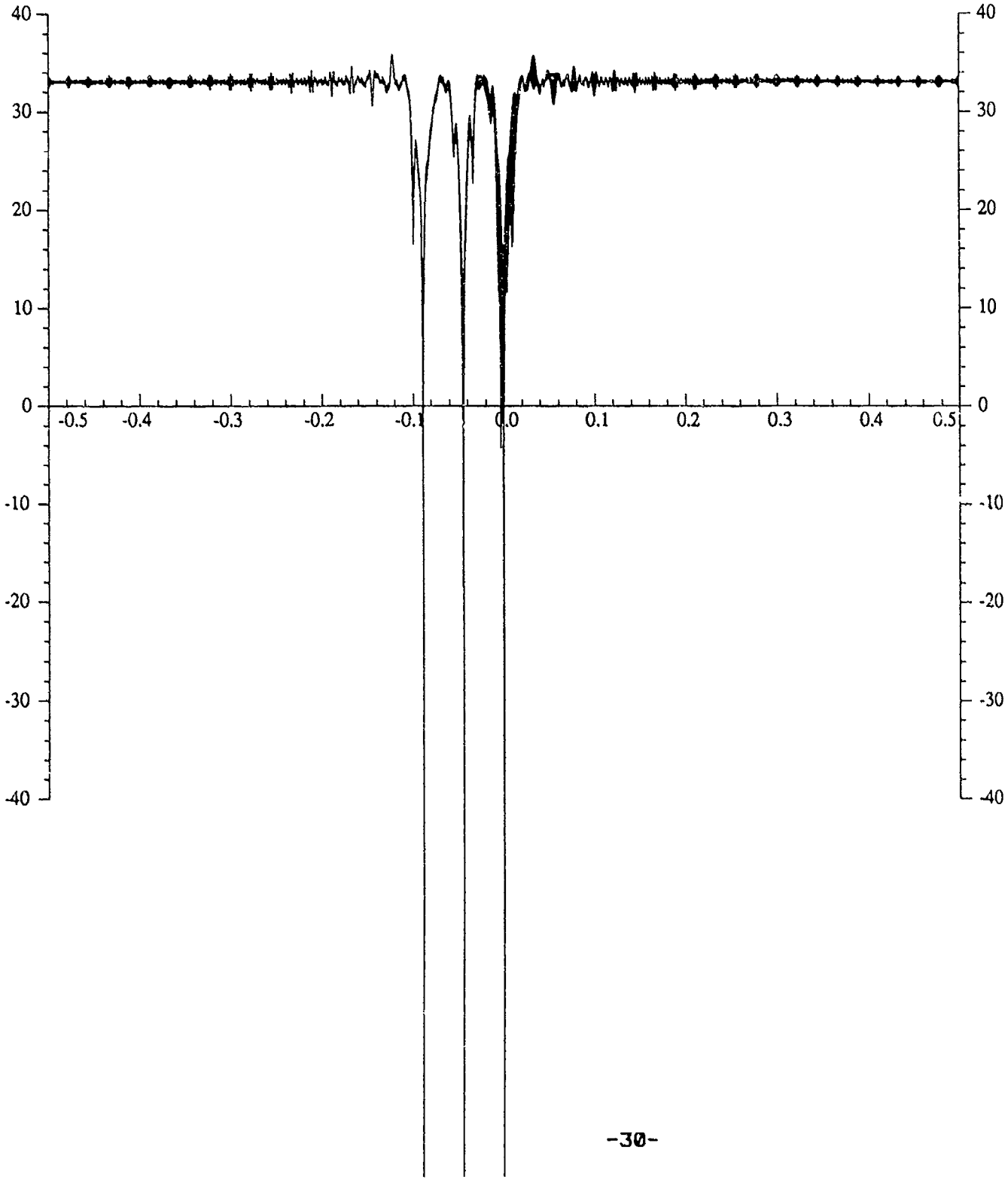
A Double Notch Filter, Degree 2024



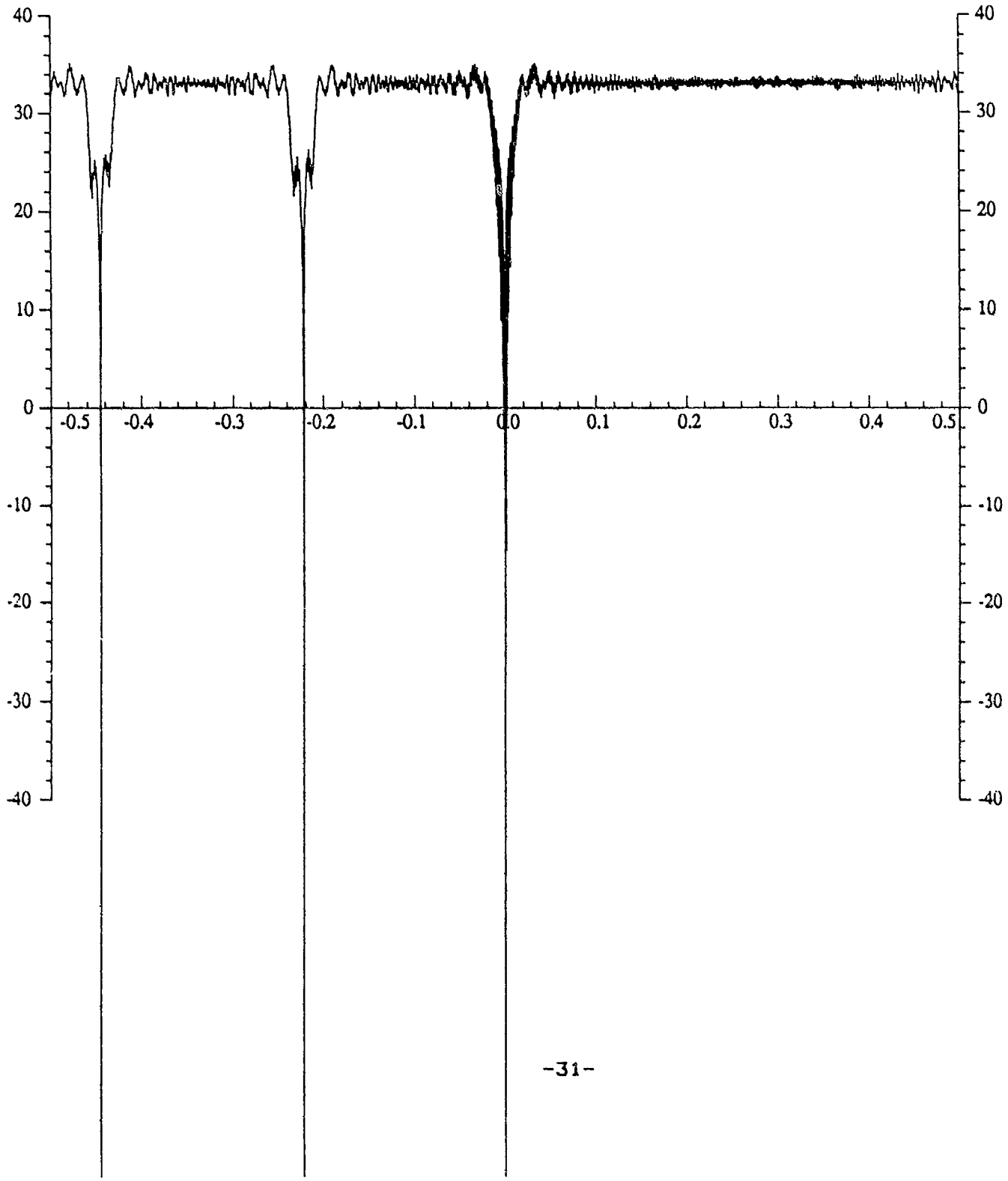
A Double Notch Filter, Degree 2024



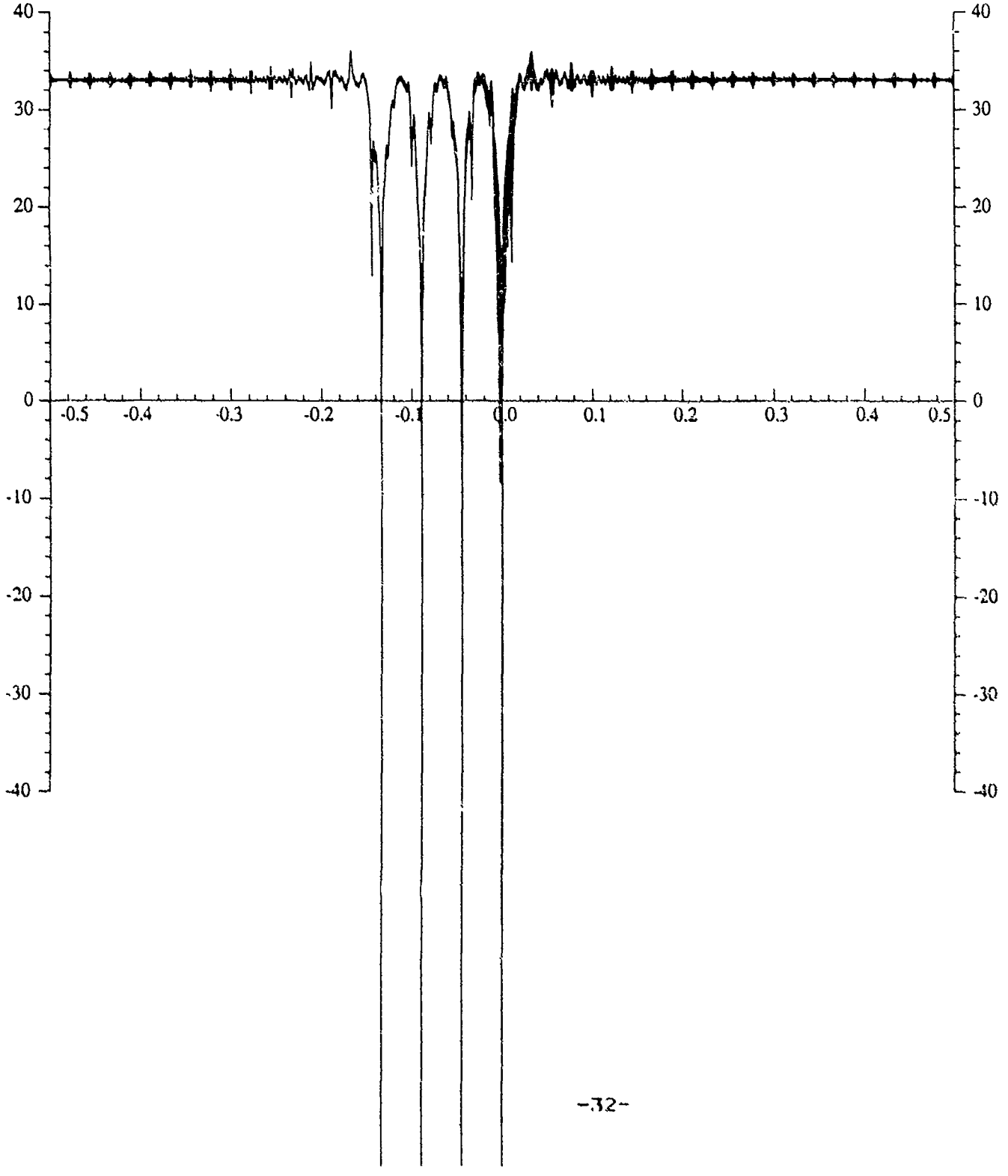
A Triple Notch Filter, Degree 2024



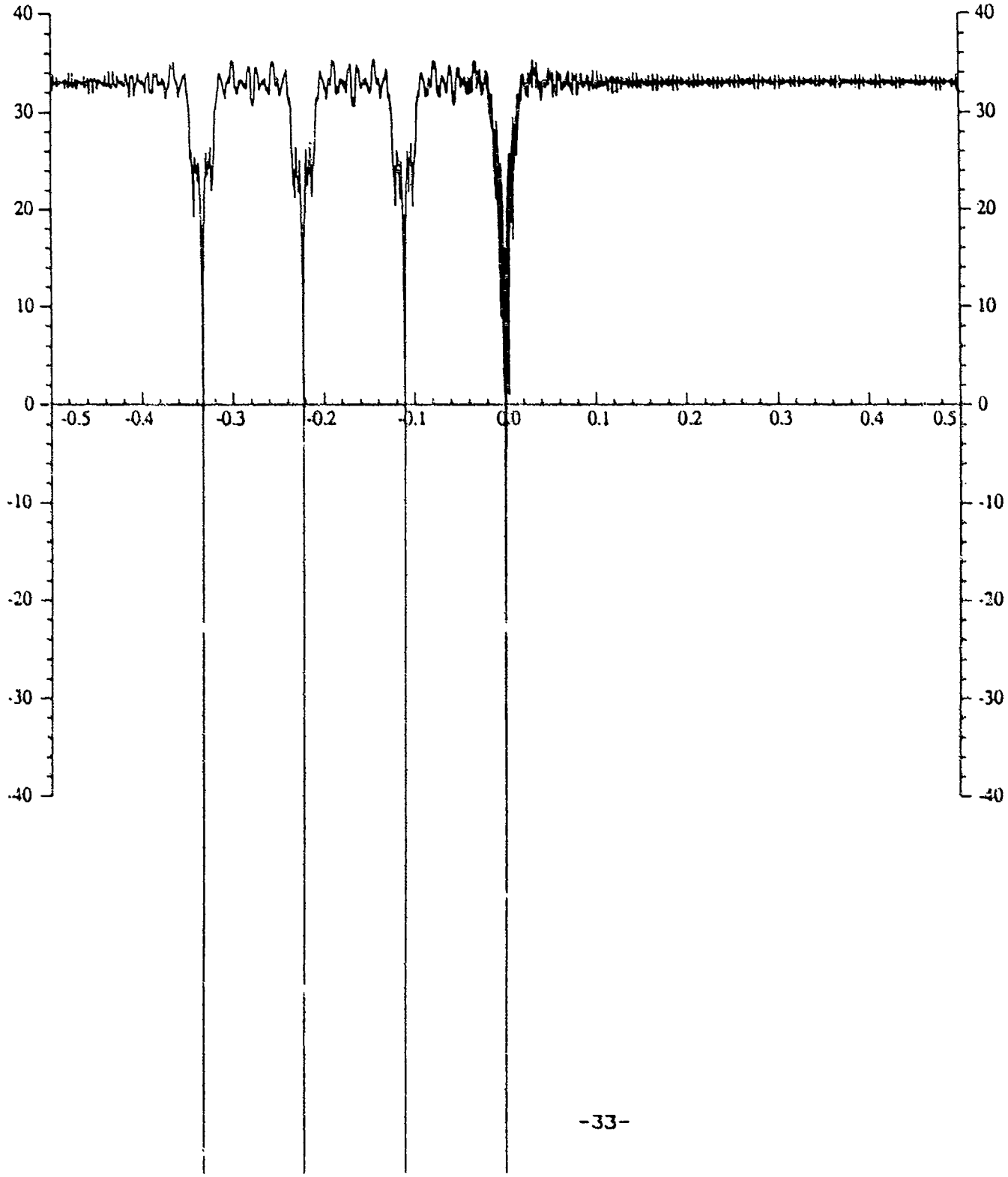
A Triple Notch Filter, Degree 2024



A Quadruple Notch Filter, Degree 2024



A Quadruple Notch Filter, Degree 2024



V. APPROXIMATION ERRORS

Another problem considered was the development of new methods of incorporating Doppler into standard models of a target moving with respect to an array. In order to make the question mathematically tractable, the simple case of a target path parallel to a passive line array of equally spaced elements was analyzed. Generalizations of the concepts described herein will be explored in Phase II.

Thus, suppose a moving target is generating a signal $s(t)$ which is being received by an N-element line array with interelement spacing d. Let $r_n(t)$ be the distance at time t between the target and the n-th array element $A_n (1 \leq n \leq N)$, y be the perpendicular distance between the target path and the line of the array, c be the speed of the signal, v be the speed of the target, and $T_n(t) = r_n(t)/c$ be the time delay from the target to A_n. Neglecting Doppler (and noise) for the moment, the received signal at A_n is $a_n(t)=s(t-T_n(t))/r_n(t)$.

The Doppler effect can be represented by a multiplicative factor u in the time variable, so that s above becomes $s(ut-T_n(ut))$. If u were constant, the required transformation of this factor into frequency domain would be simple, namely, the Fourier transform $S(w)$ of $s(t)$ would become $(1/u)S(w/u)$. However, in actuality, u is a function of both n and t, $u=1+\dot{r}_n/c$. Here, $\dot{r}_n$ represents the rate of change of $r_n(t)$ with respect to time and is negative when the target is approaching A_n. One approximate yet reasonable method of dealing with the variable Doppler is to assume that for each n, u changes exactly once as the target moves, with this change occurring at the time t_o when the target is exactly opposite the middle array element.

Letting x(t) be the target location at time t and n'=[(N+1)/2], the situation is given in figure 1.

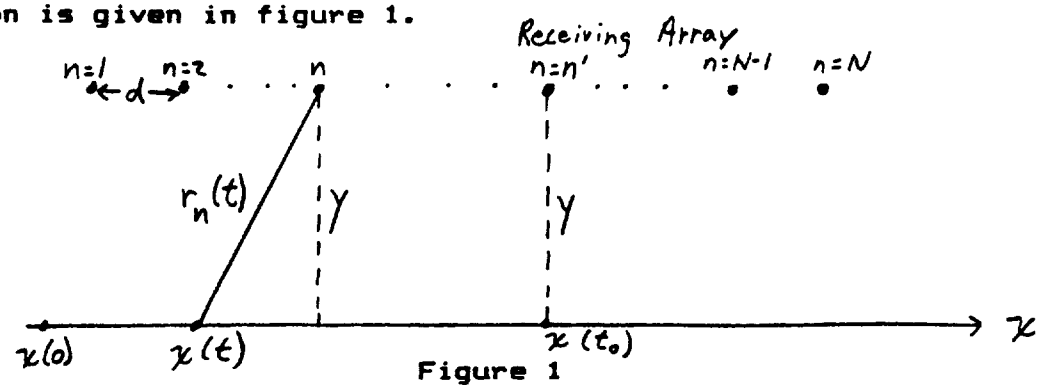


Figure 1

From this figure, clearly

$x(t)=x(0)+vt$ and

$r_n^2(t)=y^2+(x(t)-x(t_o)-(n-n')d)^2$.

To obtain a potentially useful estimate of $\dot{r}_n$, so as to estimate u, let $\delta=-1$ if $t<t_o$, $\delta=+1$ if $t\geq t_o$, τ be the length of the time interval over which the signal is measured, and r be the average value of all $r_n(t)$,

$$r=\frac{1}{N}\sum_{n=1}^{N}\frac{1}{\tau}\int_0^{\tau} r_n(t)dt.$$

Applying the Mean Value Theorem to the above equation for $r_n(t)$ yields

$$\frac{\dot{r}_n-\dot{r}_{n'}}{v}=\sqrt{1-\frac{y^2}{r_n^2}}-\sqrt{1-\frac{y^2}{r_{n'}^2}}=\frac{y^2}{2\sqrt{1-\frac{y^2}{r_\eta^2}}}\left(\frac{1}{r_{n'}^2}-\frac{1}{r_n^2}\right),$$

where $r_\eta^2=y^2+(x(t)-x(t_o)-(\eta-n')d)^2$ and η is between n and n'.

Approximating η by (n+n')/2 and using

$r_n^2-r_{n'}^2=(n-n')d\left((n-n')d-2(x(t)-x(t_o))\right)$ gives

$$\frac{\dot{r}_n-\dot{r}_{n'}}{v}=\frac{1}{2}(n-n')d\,\frac{1}{r_{n'}^2 r_n^2}\,\frac{(n-n')d-2(x(t)-x(t_o))}{x(t)-x(t_o)-(\eta-n')d}\approx\frac{y^2}{r^3}(n'-n)d.$$

Since $r_{n'}^2(t)=y^2+(x(t)-x(t_o))^2$,

$$\dot{r}_{n'}=\frac{x(t)-x(t_o)}{r_{n'}}\,v=\delta\sqrt{1-\frac{y^2}{r_{n'}^2}}\,v\approx\delta\sqrt{1-\frac{y^2}{r^2}}\,v.$$

Combining the above, we see that

$$\dot{r}_n \approx \delta\sqrt{1-\frac{d^2}{r^2}}\,v + \frac{d^2}{r^3}(n'-n)dv.$$

Finally, letting u^* be our approximation for u, so that

$$u^* = 1+\frac{v}{c}\left[\delta\sqrt{1-\frac{d^2}{r^2}} + \frac{d^2}{r^3}(n'-n)d\right],$$

the Fourier transform of s(t) now becomes $(1/u^*)S(w/u^*)$, and this quantity is employed in the processing of the received signal.

New methods of accounting for the effect of wavefront curvature upon beamforming were also considered. A preliminary result is given below, and this effort will continue in Phase II.

Referring to figure 2, the receiving array is again a line array of N equally spaced elements, with interelement spacing d. The distance from a particular source to the n-th receiving element is denoted by $r+x_n$ (with $x_1=0$), and ϕ is the indicated angle from the first array element to the source.

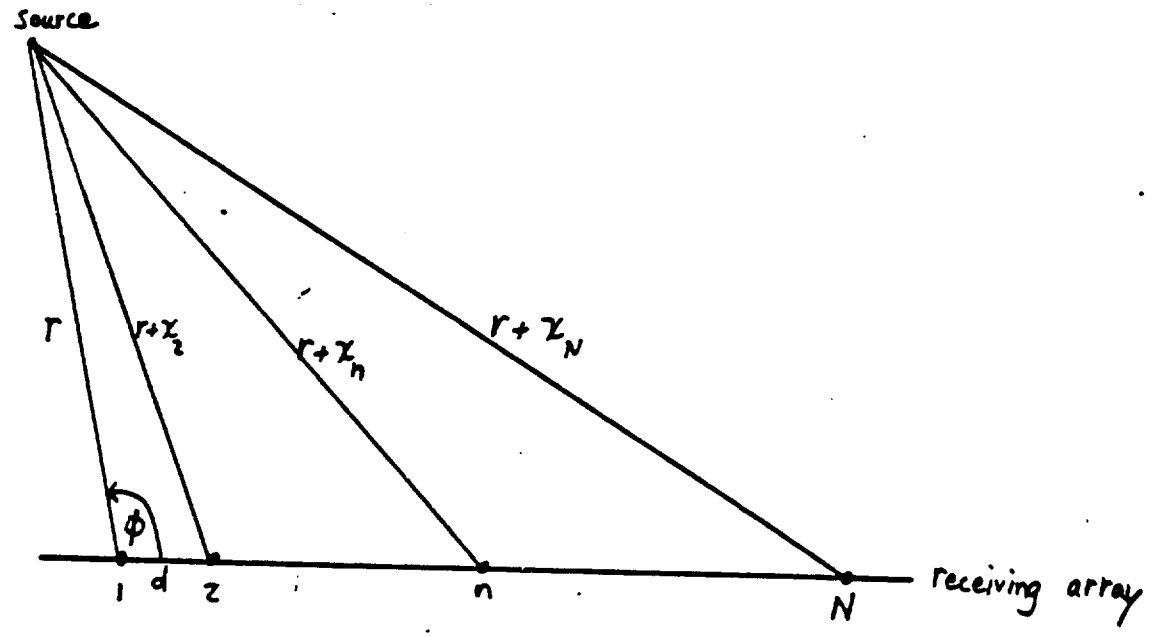


Figure 2

$$(r+x_n)^2 = r^2 + ((n-1)d)^2 - 2r(n-1)d\cos\phi. \qquad (1)$$

Assuming that the array length$<<r$ and neglecting all terms beyond the first order, (1) yields

$$x_n = ((n-1)d)^2/2r - (n-1)d\cos\phi. \qquad (2)$$

Letting R_n be the voltage response of the n-th element, and normalizing so that $R_1=1$, it follows that $R_n = r/(r+x_n)$. Now, with v_n denoting the output voltage and a_n the shading coefficient of the n-th element, and assuming a sinusoidal input of frequency ω, it follows that the output voltage of the array is

$$V = \sum_{n=1}^{N} a_n v_n = \sum_{n=1}^{N} a_n (r/(r+x_n))\exp(j\omega t)\exp(2\pi j x_n/\lambda). \qquad (3)$$

Here λ is the wavelength of the source, so that $2\pi x_n/\lambda$ is the phase delay from the first to the n-th array element.

In most cases values for V will be obtained computationally, as a closed form for the sum in (3) is usually impossible to obtain. However, if the shading coefficients a_n are all 1, and if it is assumed that the x_n increase linearly from x_1 to x_N, then the closed form solution for V is

$$V = \exp(j\omega t)\left\{\frac{1-e^{j\frac{\pi}{\lambda}\gamma_N N}}{1-e^{j\frac{\pi}{\lambda}\gamma_N}} - \frac{e^{j\frac{\pi}{\lambda}\gamma_N(N+1)}}{r\left(1-e^{j\frac{\pi}{\lambda}\gamma_N}\right)}\left(\frac{\sin\frac{\pi}{\lambda}\gamma_N N}{\sin\frac{\pi}{\lambda}\gamma_N} - Ne^{j\frac{\pi}{\lambda}\gamma_N(N-1)}\right)\right\}, \qquad (4)$$

where $\gamma_N = (N-1)d^2/2r - d\cos\phi$.

To derive (4), observe that the assumption that x_n increases linearly from x_1 (=0) to x_N, when combined with (2), yields

$$x_n = \frac{n-1}{N-1} x_N \approx (n-1)\left(\frac{(N-1)d}{2r} - d\cos\phi\right) = (n-1)\gamma_N , \qquad (5)$$

with γ_N defined as in (4). From (3), the fact that all $a_n = 1$, and the assumption that array length<<r, follows

$$v_n = \frac{r}{r+(n-1)\gamma_N} e^{j\frac{2\pi}{\lambda}(n-1)\gamma_N} e^{j\omega t} \approx \left(1-(n-1)\frac{\gamma_N}{r}\right) e^{j\frac{2\pi}{\lambda}(n-1)\gamma_N} e^{j\omega t} . \qquad (6)$$

Defining

$$g(u) = \sum_{n=1}^{N} e^{j2\pi\gamma_N u(n-1)} = \frac{1-e^{j2\pi\gamma_N uN}}{1-e^{j2\pi\gamma_N u}} , \qquad (7)$$

it follows from (6) and the termwise differentiation of (7) that

$$V = \sum_{n=1}^{N} v_n = e^{j\omega t}\left(g\left(\frac{1}{\lambda}\right) + \frac{j}{2\pi r} g'\left(\frac{1}{\lambda}\right)\right). \qquad (8)$$

Since

$$g'(u) = \frac{2\pi j \gamma_N e^{\pi j \gamma_N u(N+1)}}{1-e^{2\pi j\gamma_N u}} \left(\frac{\sin(\pi\gamma_N uN)}{\sin(\pi\gamma_N u)} - Ne^{\pi j\gamma_N u(N-1)}\right),$$

(4) is an immediate consequence of (6), (7), and (8).

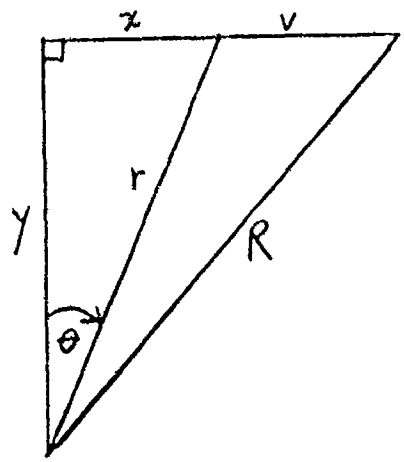


Figure 3. Configuration for the Fresnel Approximation

Next we analyze the error in the Fresnel Approximation which, for R-r in figure 3, is

$$R-r \approx v \sin\theta + (v^2\cos^2\theta)/2r.$$

This approximation, useful when $v \ll r$ (and even more useful when θ is small), arises by truncating the power series expansion of the square root in the equation

$$R = r\sqrt{1 + \frac{v^2}{r^2} + \frac{2v}{r}\sin\theta}.$$

To estimate the error in this approximation, we require more detail in the diagram, as in figure 4.

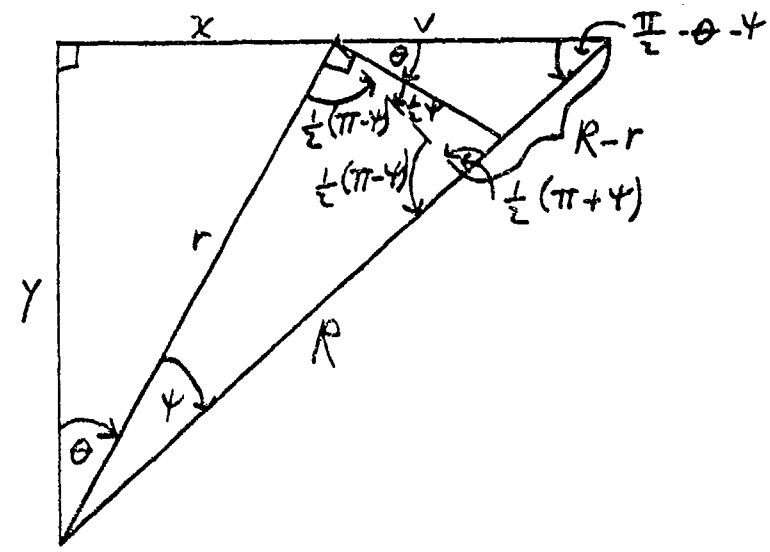


Figure 4

By the law of sines,

$$v/\sin((\pi+\psi)/2) = (R-r)/\sin(\theta+\psi/2), \text{ or}$$

$$v/\cos(\psi/2) = (R-r)/(\sin\theta\cos\psi/2 + \cos\theta\sin\psi/2), \text{ so}$$

$$R-r = v\sin\theta + v\cos\theta\tan\Psi/2 = v\sin\theta + v\cos\theta(1-\cos\Psi)/\sin\Psi. \quad (*)$$

From the law of cosines, $\cos\Psi = (r^2+R^2-v^2)/2rR$, and from the law of sines, $v/\sin\Psi = R/\sin(\theta+\pi/2) = R/\cos\theta$, so $\sin\Psi = v\cos\theta/R$. Inserting these in the expression for R-r, we get

$$R-r = v\sin\theta + v\cos\theta\,(R/v\cos\theta)\left(1-\frac{r^2+R^2-v^2}{2rR}\right)$$

$$= v\sin\theta + R - \frac{r^2+R^2-v^2}{2r} = v\sin\theta + \frac{v^2-(R-r)^2}{2r}.$$

Denoting the desired error by E, we have

$$E = R-r-v\sin\theta - v^2\cos^2\theta/2r$$

$$= R-r-v\sin\theta - \frac{v^2-(R-r)^2}{2r} + \frac{v^2-(R-r)^2}{2r} - v^2\cos^2\theta/2r$$

$$= v^2(1-\cos^2\theta-(R-r)^2/v^2)/2r$$

$$= \frac{v^2}{2r}\left(\sin\theta - \frac{R-r}{v}\right)\left(\sin\theta + \frac{R-r}{v}\right)$$

$$= \frac{v^2}{2r}(-\cos\theta\tan\Psi/2)(2\sin\theta + \cos\theta\tan\Psi/2,$$

where the last equality follows from (*). Our estimate comes from this last expression. Observe that for fixed v, the maximum of $\tan\Psi/2$ will occur when $\theta = -\Psi/2$, i.e., where the situation is as in figure 5.

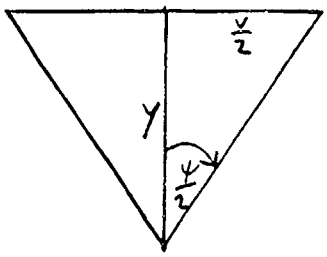


Figure 5.

Thus, $|\tan\Psi/2| \leq v/2y$, so that

$$|E| \leq \frac{v^3}{4y^2}\left(2\sin|\theta_{max}| + \frac{v}{2y}\right).$$

Also, for small enough θ, we certainly have $|\sin\theta| \leq v/y$, so the final estimate for this Fresnel Approximation error becomes

$$|E| \leq 5v^4/(8y^3).$$

We now obtain a similar estimate for the near-normal Fresnel Approximation. Again referring to figure 3 above, this approximation is

$$R-r \approx ((x+v)^2-x^2)/2y = xv/y + v^2/2y.$$

It arises by differencing the truncated power series expansions of the square roots in

$$R=y\sqrt{1+((x+v)/y)^2},\quad r=y\sqrt{1+(x/y)^2}.$$

Since near-normal incidence is being considered here, to estimate the error we begin by assuming that $|x|<v$. An exact equation for R-r, also given above, is (assuming $\theta>0$)

$$R-r = xv/r + (v^2-(R-r)^2)/2r.$$

Thus, the error which we wish to estimate, E*, is

$$E^* = xv(1/y-1/r)+v^2(1/y-1/r)/2+(R-r)^2/2r$$

Note that $E^*>0$, so that the exact value of R-r is less than the

approximate value,

$$R-r<xv/y+v^2/2y<3v^2/2y.$$ Also,

$$r-y=x^2/(r+y)<x^2/2y.$$

Putting these estimates into the above expression for E* yields

$$0<E^*<x^3v/2ry^2 + x^2v^2/4ry^2 + 9v^4/8ry^2.$$

Since x<v and r>y,

$$0<E^*<15v^4/8y^3.$$

The presence of errors resulting from such causes as clutter scatterer returns, element thermal noise, electronic counter^measures, antenna location errors, and multipath effects often gives rise to instability in nonlinear high-resolution beamforming techniques. One reason is that these methods can place undue emphasis upon unstable eigenvectors of the noise covariance matrix [11]. Hence the question of how errors in the entries of the noise covariance matrix effect the basic properties of an array, such as its gain, is of interest. Preliminary results in this regard may be found in Byrnes and Sullivan [19], and a continuation and expansion of these ideas is presented in Appendix F.

Beamforming enables an array to act as a spatial filter, by enhancing detectability, resolution, and directional measurement of plane wave signals. There are certain applications, however, where beamforming can offer only peripheral assistance, and other methods are required. An important example in this regard is low-frequency detection in the ocean. Propagation loss modeling is a fundamental tool applied to the detection of targets at medium to long ranges, and the Parabolic Equation Method (PEM) has recently become a basic technique in the attempt to improve the solvability and accuracy of such models. Our investigations into the PEM, including the discovery of certain inaccuracies in its derivation, are described in Appendix C.

VI. ROBUSTNESS

We now report on additional progress in determining the robustness of trigonometric polynomials with unimodular coefficients. As indicated in [18], we do this by estimating the maximum of trigonometric polynomials with random coefficients. These random polynomials represent the deviation in frequency response of a linear, equispaced antenna array caused by coefficient inaccuracy in the basic unimodular polynomial of the array.

Let us consider for this exposition the random cosine polynomial

$$P(t,\omega) = \sum_{k=1}^{n} X_k(\omega) \cos kt,$$

where the X_k are independent, standard normal random variables, denoted X_k i.i.d $N(0,1)$. Our results apply immediately to the sine polynomial $\sum_{k=1}^{n} X_k \sin kt$ and may also be generalized to the random trigonometric polynomial $\sum_{k=-n}^{n} X_k e^{ikt}$.

We define

$$M(\omega) = \sup_{t\in[0,2\pi]} |P(t,\omega)|$$

and we wish to estimate $P\{\alpha_1 < M \leq \alpha_2\}$. Note that n, the degree of $P(t,\omega)$, is a fixed, positive integer. Simple upper bounds for $P\{M > \alpha\}$ were obtained in our earlier work, and we now seek a lower bound for this probability to complete our estimation. In the future, we hope to expand this work to include more general random coefficients. For now, we limit ourselves to the normal case.

We give two results. The first is less precise and less useful than the second, but we include it to introduce a method of promising simplicity which we hope to exploit later. The second is a

generalization of a result of Salem and Zygmund [38] to our case. After stating and proving these results, we shall compare their precision with other similar estimates.

Proposition 1: (1) $P\{M>\beta\} \geq 1 - \frac{\beta e^{(\frac{1}{2}-\frac{\beta^2}{n})}}{\sqrt{n/2}}$ where $0<\beta<\sqrt{n/2}$,

or equivalently

(2) $P\{M\leq\beta\}\leq \frac{\beta e^{(\frac{1}{2}-\frac{\beta^2}{n})}}{\sqrt{n/2}}$ where $0<\beta<\sqrt{n/2}$.

Proposition 2: Let $\delta\in(0,1)$. Then

$$(1)\quad P\{M\geq \sqrt{n\log n} + \frac{\log\delta}{4}\sqrt{\frac{n}{\log n}}\} \geq \frac{(1-\delta)^2}{1+\frac{5}{36}n^{24}}$$

In particular, if $\delta = (\log n)^{-1}, n\geq 3$, we have

$$(2)\quad P\{M\geq \sqrt{n\log n} - \frac{\log\log n}{4}\sqrt{\frac{n}{\log n}}\} \geq \frac{[1-(\log n)^{-1}]^2}{1+\frac{5}{36}n^{24}}$$

Proof of Proposition 1:

We prove Proposition 1 (2).

Let $\lambda>0$. $P\{M\leq\beta\}=P\{-\frac{\lambda}{2}M^2\geq-\frac{\lambda}{2}\beta^2\}=P\{e^{-\frac{\lambda}{2}M^2}\geq e^{-\frac{\lambda}{2}\beta^2}\}$.

By Markov's Inequality,

$$P\{M\leq\beta\}\leq e^{\frac{\lambda}{2}\beta^2}E\left(e^{-\frac{\lambda}{2}M^2}\right). \qquad (9)$$

Evidently, $M^2(\omega)\geq P^2(t,\omega)$ for every $t\in[0,2\pi]$, and since

$P(t,\omega)\sim N(0,\sum_{k=1}^{n}\cos^2 kt)$, we have

$\frac{P(\hat{\theta}, \omega)}{\sum_{k=1}^{n} \cos^2 kt} \sim \chi_1^2$ (chi-square with one degree of freedom).

Therefore, using the fact that the moment generating function ϕ of a χ_1^2 random variable is $\phi(\xi) = 1/\sqrt{1-2\xi}$ for $\xi < 1/2$.

$$E(e^{-\frac{\lambda}{2}M^2}) \le E(e^{-\frac{\lambda}{2}P^2}) = E\left(e^{-\frac{\lambda}{2}\sum_{k=1}^{n}\cos^2 kt\,\chi_1^2}\right) \qquad (10)$$

$$= \frac{1}{\sqrt{1+\frac{\lambda}{2}\sum_{k=1}^{n}\cos^2 kt}}.$$

Combining (9) and (10) yields

$$P(M \le B) \le \frac{e^{\frac{\lambda}{2}B^2}}{\sqrt{1+\frac{\lambda}{2}\sum_{k=1}^{n}\cos^2 kt}}.$$

Now let us find $\lambda > 0$ and $t \in [0, 2\pi]$ which minimize the right side of the above inequality. Clearly, $t=0$ is the correct value giving $\sum_{k=1}^{n} \cos^2 k \cdot 0 = n$ and thus

$$P\{M \le B\} \le \frac{e^{\frac{\lambda}{2}B^2}}{\sqrt{1+\frac{\lambda}{2}n}} = h(\lambda). \qquad (11)$$

To find λ which minimizes $h(\lambda)$, we have

$$h'(\lambda) = \frac{e^{\frac{\lambda}{2}B^2}\left[\frac{B^2}{2}\left(1+\frac{\lambda}{2}n\right) - \frac{n}{4}\right]}{\left(1+\frac{\lambda}{2}n\right)^{3/2}}$$

Thus, $h'(\lambda) = 0 \iff \frac{B^2}{2}\left(1+\frac{\lambda}{2}n\right) - \frac{n}{4} = 0 \iff \lambda = \frac{1}{B^2} - \frac{2}{n}$,

and λ is obviously a minimum point of h. To insure $\lambda > 0$, we require $B < \sqrt{\frac{n}{2}}$ and substituting the critical value of λ in equation (11), we have

$$P\{M \le B\} \le \frac{Be^{\frac{1}{2} - \frac{B^2}{n}}}{\sqrt{n/2}}$$ giving the result.

Proof of Proposition 2:

We follow the method in Salem and Zygmund [38], modifying their results to fit our case. We begin with a preliminary lemma.

Lemma 1: Let $Q(\omega)$ be a non-negative random variable satisfying $E(Q)\geq A>0$ and $E(Q^2)\leq B$ where E denotes the mathematical expectation. Then, if $\delta\in(0,1)$, $P\{Q\geq\delta A\}\geq(1-\delta)^2A^2/B$.

proof

Put $\mathcal{D}=\{\omega:Q(\omega)\geq\delta A\}$. Then $\int_{\mathcal{D}^c} Q(\omega)dP(\omega)\leq\delta AP(\mathcal{D}^c)\leq\delta A$

$$\int_{\mathcal{D}} QdP = \int_{\Omega}QdP - \int_{\mathcal{D}^c} QdP \geq EQ-\delta A \geq A-\delta A = (1-\delta)A.$$

Also, $\int_{\mathcal{D}} QdP \leq \left(\int_{\mathcal{D}} 1^2dP\right)^{1/2}\left(\int_{\mathcal{D}} Q^2dP\right)^{1/2}$

$\leq P(\mathcal{D})^{1/2}B^{1/2}$.

Combining the above inequalities, we have $(1-\delta)A\leq P(\mathcal{D})^{1/2}B^{1/2}$, which yields the conclusion of the lemma.

We wish to apply lemma 1 to the random variable

$$I(\omega)=\frac{1}{2\pi}\int_0^{2\pi} e^{\lambda P(t,\omega)}dt.$$

To do this, we must verify that $I(\omega)$ satisfies the hypotheses of the lemma, and we must obtain the constants A and B.

By Tonelli's theorem,

$$E(I) =\frac{1}{2\pi}\int_0^{2\pi} E(e^{\lambda P(\cdot,\omega)})dt =\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\lambda^2}{2}\sum_{k=1}^{n}\omega_k^2}dt$$

using the form of the moment generating function of $P(t,\omega)$ already obtained. With the simple identity $\cos^2 kt = \frac{1}{2}[1+\cos 2kt]$, we find

$$E(I) = \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\lambda^2}{4}n}\; e^{\frac{\lambda^2}{4}\sum_{k=1}^{n}\cos 2kt}\,dt$$

$$\geq e^{\frac{\lambda^2}{4}n}\,\frac{1}{2\pi}\int_0^{2\pi}\left(1+\frac{\lambda^2}{4}\sum_{k=1}^{n}\cos 2kt\right)dt = e^{\frac{\lambda^2}{4}n}.$$

So we have $E(I) \geq e^{\frac{\lambda^2}{4}n} = A$ in lemma 1.

We now require a number B such that $E(I^2)\leq B$ to satisfy the second hypothesis of lemma 1. To do this, we write

$$I^2(\omega) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} e^{\lambda[P(t,\omega)+P(\tau,\omega)]}\,dt\,d\tau.$$

Then $E(I^2) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} E\left(e^{\lambda[P(t,\omega)+P(\tau,\omega)]}\right)dt\,d\tau$,

using Tonelli's theorem.

$$P(t,\omega) + P(\tau,\omega) = \sum_{k=1}^{n} X_k(\omega)\,(\cos kt + \cos k\tau)$$

$$\sim N(0, \sum_{k=1}^{n}(\cos kt + \cos k\tau)^2)$$

because the sum of independent normal random variables is normal. We thus have that the moment generating function of $P(t,\omega) + P(\tau,\omega)$ is

$$E\left(e^{\lambda[P(t,\omega)+P(\tau,\omega)]}\right) = e^{\frac{\lambda^2}{2}\sum_{k=1}^{n}(\cos kt+\cos k\tau)^2}.$$

Then

$$E(I^2) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} e^{\frac{\lambda^2}{2}\sum_{k=1}^{n}(\cos kt+\cos k\tau)^2}\,dt\,d\tau.$$

Now $(\cos kt + \cos k\tau)^2 = 1+\frac{1}{2}\cos 2kt + \frac{1}{2}\cos 2k\tau + 2\cos kt\cos k\tau$.

Put $S = S(t,\tau) = \sum_{k=1}^{n}\left(\frac{1}{2}\cos 2kt + \frac{1}{2}\cos 2k\tau + 2\cos kt\cos k\tau\right)$.

This gives

$$E(I^2) = e^{\frac{\lambda^2}{2}n} \frac{1}{4\pi^2} \int_0^{2\pi}\int_0^{2\pi} e^{\frac{\lambda^2 S}{2}} \, dt \, d\tau. \qquad (12)$$

To bound the final integral, we will use lemma 4.2.2 of Salem and Zygmund which we state for functions of two variables.

Lemma 2 (Salem and Zygmund)

Let $g(x,y)$, $a \le x \le b$, $c \le y \le d$, be a bounded real function. Suppose that $|g(x,y)| \le A$ and $(b-a)^{-1}(d-c)^{-1} \int_a^b \int_c^d g^2(x,y)dxdy = B$. Then, for any positive number u,

$$(b-a)^{-1}(d-c)^{-1} \int_c^d \int_a^b e^{ug(x,y)}dxdy \le 1+u\sqrt{B}+\frac{B}{A^2}e^{uA}.$$

In the case when $\int_c^d \int_a^b g(x,y)dxdy = 0$, this inequality can be replaced by

$$(b-a)^{-1}(d-c)^{-1} \int_c^d \int_a^b e^{ug(x,y)}dxdy \le 1+\frac{B}{A^2}e^{uA}.$$

We apply lemma 2 to the function $S=S(t,\tau)$ for $0 \le t \le 2\pi$, $0 \le \tau \le 2\pi$. To verify the necessary hypotheses, first note that

$$\int_0^{2\pi} \int_0^{2\pi} S(t,\tau)dtd\tau = 0.$$

The set of 3n functions $\{\cos 2kt, \cos 2k\tau, \cos 2kt \cos 2k\tau : k=1,\ldots n\}$ is an orthogonal system over the square of integration. This leads to

$$\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} S^2(t,\tau)dt \, d\tau$$

$$= \frac{1}{4\pi^2} \sum_{k=1}^{n} \int_0^{2\pi} \int_0^{2\pi} \left(\frac{1}{4} \cos^2 2kt + \frac{1}{4} \cos^2 2k\tau + 4\cos^2 kt \cos^2 k\tau\right) dt\,d\tau$$

$$= \left(\frac{1}{8} + \frac{1}{8} + 1\right) n = \frac{5}{4} n = B \text{ of lemma 2.}$$

Evidently, $|S(t,\tau)| \leq 3n = A$ of lemma 2. Thus, the hypotheses of lemma 2 are satisfied and we obtain, with $u = \lambda^2/2$,

$$\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{\frac{\lambda^2}{2} S(t,\tau)} \, dt\,d\tau \leq 1 + \frac{(5/4)n}{9n^2} e^{\frac{3}{2}\lambda^2 n}.$$

Applying this to equation (12) gives

$$E(I^2) \leq e^{\frac{\lambda^2}{2} n} \left[1 + \frac{5}{36n} e^{\frac{3}{2}\lambda^2 n}\right] = B \text{ of lemma 1.}$$

Recall that we have obtained $E(I) \geq e^{\frac{\lambda^2}{4} n} = A$ of lemma 1. Thus, we apply lemma 1 to obtain

$$P\left\{I \geq \delta e^{\frac{\lambda^2}{4} n}\right\} \geq \frac{(1-\delta)^2}{1 + \frac{5}{36n} e^{\frac{3}{2}\lambda^2 n}} \quad \text{for } \delta \in (0,1). \tag{13}$$

To prove proposition 2, we note that for every $\lambda > 0$, $t \in [0, 2\pi]$ and ω, we have $e^{\lambda P(t,\omega)} \leq e^{\lambda M(\omega)}$. Applying $\frac{1}{2\pi} \int_0^{2\pi} dt$ to both sides of the inequality yields $I(\omega) \leq e^{\lambda M(\omega)}$ for each $\lambda > 0$. If we use this with (13), we get

$$P\left\{M \geq \frac{1}{\lambda} \log\delta + \frac{\lambda}{4} n\right\} = P\left\{e^{\lambda M} \geq \delta e^{\frac{\lambda^2}{4} n}\right\}.$$

$$\geq P\left\{I \geq \delta e^{\frac{\lambda^2}{4} n}\right\} \geq \frac{(1-\delta)^2}{1 + \frac{5}{36n} e^{\frac{3}{2}\lambda^2 n}}.$$

Now put $\lambda = 4\sqrt{\frac{\log n}{n}}$ in the inequality to obtain proposition 2(1) and then substitute $\delta = (\log n)^{-1}$, $n \geq 3$, to obtain proposition 2(2).

Discussion of Results

We now examine propositions 1 and 2 and compare them to other similar estimates. An excellent standard of comparison is the recent asymptotic result of Turkman and Walker [44] that

$$\lim_{n \to \infty} P\left\{ \frac{M - \sqrt{n \log n} + \frac{\log 12}{4}\sqrt{\frac{n}{\log n}}}{\frac{1}{2}\sqrt{\frac{n}{\log n}}} \leq x \right\} = \exp[-e^{-x}]$$

for every real number x. Roughly, this says that for large n, $M = M_n$ is centered near $\sqrt{n \log n}$ with dispersion of the order of $\sqrt{\frac{n}{\log n}}$. The difficulty in applying this result is that, for a given n, it is not known how close the distribution of

$$\frac{M - \sqrt{n \log n} + \frac{\log 12}{4}\sqrt{\frac{n}{\log n}}}{\frac{1}{2}\sqrt{\frac{n}{\log n}}}$$

is to the limiting distribution.

Comparing this result to Proposition 2(2) shows that our centering constants closely resemble those of Turkman and Walker and suggests that the bounds obtained in Proposition 2 are indeed tight. In addition, the lower bound of Proposition 2(2) is better than a similar bound obtained by Gersho, Gopinath and Odlyzko [22] in the second part of their theorem. The Gersho bound, however, applies to a more general class of random trigonometric polynomials. It is obvious that the bounds of Proposition 2 are much superior to those of

Proposition 1, which applies only when $\beta < \sqrt{n/2}$, which is far below the central values of M. Proposition 1 has been included only to illustrate a simple but promising method which we hope to exploit in the future.

The future program is to generalize the methods of Proposition 2 to trigonometric polynomials with independent random coefficients which are not necessarily normally distributed. Of particular interest is the case of uniformly distributed coefficients which represent digital or round-off errors. While the applicability of the methods of Salem and Zygmund to the attainment of an upper bound for $P(M > \alpha)$ has been noted by Gersho, et al and others, no one, to our knowledge, has used these methods to obtain a lower bound as we have done in Proposition 2. This development offers hope for improved probablistic estimates of the maximum of trigonometric polynomials with general random coefficients.

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APPENDIX A
Additional Information

A-I. TECHNICAL PUBLICATIONS

1. Newman, D.J. and Byrnes, J.S., The L^{∞} Norm of a Polynomial with Coefficients ± 1 (revised version), American Mathematical Monthly (to appear).

2. Byrnes, J.S. and Newman, D.J., Null Steering Employing Polynomials with Restricted Coefficients (revised version), IEEE Transactions on Antennas and Propagation (to appear February, 1988)

3. Byrnes, J.S., A Notch Filter Employing Coefficients of Equal Magnitude (revised version), (submitted to IEEE Transactions on Acoustics, Speech and Signal Processing).

4. Byrnes, J.S., An Analysis of the Parabolic Equation Method and its Applications, (submitted to IEEE Journal of Oceanic Engineering).

5. Byrnes, J.S., Error Estimates Resulting from the Norms of Certain Noise Covariance Matrices, (to be submitted to the Journal of Sound and Vibration).

A-II. PROFESSIONAL PERSONNEL

1. James S. Byrnes, Principal Investigator. President, Prometheus Inc. Ph.D. (Mathematics), Yeshiva University, 1967.

2. Donald J. Newman, Principal Scientist. Ph.D. (Mathematics), Harvard University, 1953.

3. Stephen Boyd, Senior Scientist. Ph.D. (Electrical Engineering), University of California at Berkeley, 1985.

4. Andre Giroux, Senior Scientist. Ph.D. (Mathematics), University of Montreal, 1973.

5. Martin Goldstein, Senior Scientist. Ph.D. (Mathematics), University of Wisconsin, 1969.

A-III. INTERACTIONS

I. Invited Papers Presented by J. S. Byrnes.

1. Naval Weapons Center, China Lakes, CA, 20 April, 1987

2. University of Maryland Department of Mathematics, 27 April, 1987

3. Royal Institute of Technology Department of Mathematics, Stockholm, 25 June, 1987

4. NATO Advanced Study Institute on Electromagnetic Modeling and Measurements for Analysis and for Synthesis Problems, Tuscany, Italy, 18 August, 1987

5. University of Montreal Mathematics Research Institute, 10 November, 1987

II. Invited Paper Presented by Stephen Boyd

1. Naval Weapons Center, China Lakes, CA, 20 April, 1987

III. Consultations on Potential Air Force and Navy Applications of Prometheus Ideas.

1. Stephen Boyd and James Byrnes consulted with Bob Dinger, Bill Altop, Gary Hewer, Francis Canning, and Mike Mumford at the Naval Weapons Center, China Lakes, CA, 20 April, 1987.

2. James Byrnes consulted with Emanuel Vegh and Bill Gabriel at the Naval Research Laboratory, Washington, DC, 24 April, 1987.

3. James Byrnes consulted with Neil Gerr and Jim Smith at the Office of Naval Research, Arlington, VA, 24 April, 1987.

4. James Byrnes consulted with George Behnke of the MITRE Corporation, 27 April, 1987 (this meeting took place at the University of Maryland).

5. James Byrnes consulted with H. S. Shapiro at the Royal Institute of Technology, Stockholm, 24-26 June, 1987.

6. James Byrnes consulted with numerous people, especially E. K. Miller of the Rockwell Science Center and the University of Kansas, N. H. Farhat of the University of Pennsylvania, Y. Rahmat-Samii of the Jet Propulsion Laboratory, Tapan Sarkar of Syracuse University, and Anton G. Tijhuis of Technische Hogeschool Delft at the NATO Advanced Study Institute on

Electromagnetic Modeling and Measurements for Analysis and for Synthesis Problems, Tuscany, Italy, 10-21 August, 1987.

7. James Byrnes and Donald Newman consulted with Bob Mailloux, Hans Steyskal and Jeff Herd at the Rome Air Development Center, Hanscomb AFB, Bedford, MA, 24 August, 1987.

8. James Byrnes consulted with Jack Wallace, Mark Godino and Mike Carpenter at Wright Patterson AFB, Dayton, OH, 25 August, 1987. This meeting was also attended by Arje Nachman of AFOSR.

9. James Byrnes consulted with Stanley Chamberlain and E. Ted Bick at Raytheon Submarine Signal Division, Portsmouth, RI, 22 September, 1987.

10. James Byrnes consulted with many SBIR officials, DOD representatives, and representatives of several large defense contractors at the High Technology Conference, Salt Lake City, UT, 17-18 November, 1987.

11. James Byrnes visited Sun Microsystems in Mountain View, CA, to evaluate the applicability of their engineering work stations to the Prometheus Inc. Phase II effort, 20 November, 1987.

APPENDIX B

The L^∞ Norm of a Polynomial With Coefficients ± 1

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The L^4 Norm of a Polynomial With Coefficients ± 1

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A classic unresolved question regarding polynomials

$$P(z)=\sum_{k=0}^{n-1}\epsilon_k(n)z^k$$

with coefficients $\epsilon_{k(n)}=\epsilon_k=\pm 1$ is whether the maximum modulus of such a polynomial on the unit circle can be $n^{1/2}+o(n^{1/2})$. As shown by Kahane [3], if complex coefficients of modulus 1 are allowed then not only is it possible for this property to be satisfied, but the minimum modulus can be $n^{1/2}+o(n^{1/2})$ as well. Specifically, Kahane proved that for any n there is a polynomial of degree n with coefficients of modulus one whose modulus everywhere on the unit circle is $n^{1/2}+O(n^{3/10}\log n)$.

Erdös [2] had conjectured the existence of a $c>0$ such that, for any polynomial P of the types described, $\|P\|_\infty>(1+c)n^{1/2}$. Clearly the Kahane result disproved this conjecture for the modulus 1 case, but the situation for coefficients ± 1 remains open. Employing an elegant construction Shapiro [6,5] demonstrated the achievability of the order of magnitude $n^{1/2}$, but the maximum modulus of the Shapiro polynomials is about $(2n)^{1/2}$.

Since the L^2 norm of any such polynomial is $n^{1/2}$, by the Parseval Theorem, and the L^p norm for any $p<\infty$ is a lower bound for the maximum modulus, it is natural to look at the L^p norm for some $p>2$. Motivated by these considerations, as well as the inherent tractability of the L^4 norm, we examine the L^4 norm of such polynomials. As one might expect, this leads to several interesting combinatorial questions. We provide answers to some of these, and conclude with a refined version of the Erdös conjecture.

Throughout the paper n will be a positive integer, $P(z)$ will denote the previously indicated polynomial of degree $n-1$ with coefficients ± 1, and z will lie on the unit circle, $z=e^{2\pi i\theta}$, $0\le\theta<1$. All integrals will be over $\theta\in[0,1]$. We begin with a Lemma.

Lemma $$\|P\|_{L^4}^4 = \sum_{\substack{j+k=l+m \\ 0\le j,k,l,m<n}} \varepsilon_j \varepsilon_k \varepsilon_l \varepsilon_m .$$

Proof $$\|P\|_{L^4}^4 = \int_0^1 |P(e^{2\pi i\theta})|^4 d\theta$$

$$= \int \left(\sum_{j=0}^{n-1} \varepsilon_j z^j\right)\left(\sum_{k=0}^{n-1} \varepsilon_k z^k\right)\left(\sum_{l=0}^{n-1} \varepsilon_l z^{-l}\right)\left(\sum_{m=0}^{n-1} \varepsilon_m z^{-m}\right)$$

$$= \int \text{constant terms.}$$

Since a constant term occurs in this product if and only if $j+k=l+m$, the result follows immediately.

Of interest is the expected value $E(\|P\|_{L^4}^4)$, if the coefficients ε_j are chosen at random.

Theorem 1 $E(\|P\|_{L^4}^4)=2n^2-n$.

Proof Clearly if exactly 3 of the indices j,k,l,m are identical, or if at least 3 of them are different, then $E(\varepsilon_j\cdot\varepsilon_k\cdot\varepsilon_l\cdot\varepsilon_m)=0$. It therefore follows from the lemma that

$$E\left(\|P\|_{L^4}^4\right) = \sum_{\substack{j=l \text{ and } k=m \\ \text{or } j=m \text{ and } k=l}} \varepsilon_j \varepsilon_k \varepsilon_l \varepsilon_m . \qquad (1)$$

For each of the $n(n-1)/2$ pairs of integers p,q, $0\le p<q<n$,

there are 4 terms appearing in (1), $\varepsilon_p\varepsilon_q\varepsilon_p\varepsilon_q$ being typical. The only other terms in (1) are ε_p^4, $0\le p<n$. Since all of these terms equal 1,

$$E(\|P\|_{L^4}^4)=n+4(n)(n-1)/2=2n^2-n,$$

completing the proof of Theorem 1.

We now observe the improvement that is achieved when this random choice is replaced by the Shapiro coefficients. Shapiro's polynomials are defined, together with his auxiliary polynomials Q, by the recurrence formulas

$$P_0(z)=Q_0(z)=1,$$
$$P_{m+1}(z)=P_m(z)+z^{2^m}Q_m(z), \qquad (2)$$
$$Q_{m+1}(z)=P_m(z)-z^{2^m}Q_m(z), \quad m\ge 0.$$

For the interested reader we point out articles by Brillhart et al [1] and Mendès-France and Tenenbaum [4]. The former shows that the Shapiro coefficients can be defined directly from the binary representation of the order of the polynomial, while the latter relates them to paper-folding sequences.

Theorem 2 If $n=2^k$ and $P(z)$ is the Shapiro polynomial of degree $n-1$, then

$$\|P\|_{L^4}^4=(4n^2-(-1)^k n)/3.$$

Proof It follows directly from (2) that

$$|P_{m+1}(z)|^2+|Q_{m+1}(z)|^2=2\left(|P_m(z)|^2+|Q_m(z)|^2\right)$$

so that, as observed by Shapiro,

$$|P_m(z)|^2+|Q_m(z)|^2=2^{m+1}. \qquad (3)$$

Now (2) and (3) yield

$$|P_{m+1}(z)|^2=2^{m+1}+2\mathrm{Re}(z^{2^m}Q_m(z)\bar{P}_m(z)). \qquad (4)$$

Next we observe that $z^{2^m} Q_m \overline{P}_m$ is composed solely of frequencies which are positive powers of z, so that it can be thought of as $Q_m \widetilde{P}_m$, where $\widetilde{P}_m$ is the "reversed" polynomial of P. Thus

$$\int |P_{m+1}|^4 = 2^{2m+2} + 4 \int (\mathrm{Re}\, Q_m \widetilde{P}_m)^2 .$$

Since $f \equiv Q\widetilde{P}$ is analytic and 0 at the origin,

$$0 = \int f^2 = \mathrm{Re}\left(\int f^2\right) = \int (\mathrm{Re}\, f)^2 - \int (\mathrm{Im}\, f)^2 ,$$

so that

$$\int (\mathrm{Re}\, f)^2 = \tfrac{1}{2} \int |f|^2 = \tfrac{1}{2} \int |Q_m|^2 |\widetilde{P}_m|^2 = \tfrac{1}{2} \int |P_m|^2 |Q_m|^2 .$$

Altogether then, we have

$$\int |P_{m+1}|^4 = 2^{2m+2} + 2 \int |P_m|^2 \left(2^{m+1} - |P_m|^2\right) \qquad (5)$$

$$= 2^{2m+3} - 2 \int |P_m|^4 .$$

The remainder of the proof is now simply induction on k. The result is obviously true for k=0, since $P_0(z)=1$. Furthermore, from (5) and the inductive hypothesis

$$\int |P_k|^4 = \frac{2^{2k+2} - (-2)^k}{3} ,$$

it follows that

$$\int |P_{k+1}|^4 = 2^{2k+3} - 2\,\frac{2^{2k+2} - (-2)^k}{3} = \frac{2^{2(k+1)+2} - (-2)^{k+1}}{3} , \text{ as required.}$$

This completes the proof of Theorem 2.

Note that Theorem 2 implies that the L^4 norm of the n-1st degree Shapiro polynomial is asymptotic to $\sqrt{n}$ times the fourth root of $4/3 \approx 1.07457\sqrt{n}$. Based upon extensive numerical evidence employing the Bose-Einstein statistics methodology of Statistical Mechanics, we conjecture that the Shapiro polynomials do not give the minimum L^4 norm among all polynomials of the same degree with coefficients ± 1, but that this minimum L^4 norm is asymptotically $\sqrt{n}$ times the fourth

root of $6/5 \approx 1.04664\sqrt{n}$. Observe that the truth of this conjecture would imply that of the Erdös conjecture mentioned earlier, with $c=(6/5)^{1/4}-1 \approx .04664$.

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APPENDIX C

An Analysis of the Parabolic Equation Method and its Applications

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Abstract

The parabolic equation (PE) method of solving the elliptic wave equation in underwater acoustics was developed to handle the case where the index of refraction n is dependent upon range as well as upon depth, and possibly azimuth. It is used extensively in low-frequency propagation loss modeling. Fundamental to the derivation of all acoustic PE's is the assumption that a certain commutator is negligibly small, and may be ignored. One purpose of this paper is to point out that the vanishing of the commutator actually implies that n is range-independent. Thus, in many cases, the original reason for the PE method is defeated when this commutator is assumed to be negligibly small. The second purpose of this paper is to describe and analyze the current state of the art in the low-frequency applications of the PE method, and to compare it to other methods of solving the acoustic wave equation.

Key words: Parabolic Equation Method, Propagation Loss Modeling

An Analysis of the Parabolic Equation Method and its Applications

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I. Introduction

Knowledge of how sound propagates in the ocean, for different environmental conditions and source/receiver configurations, is necessary in order to achieve optimum sonar design. The estimation of the spatial properties of the sound pressure field, as a function of source frequency, is the goal of ocean acoustic modeling. To perform this modeling effectively, the acoustic loss mechanisms in the ocean must be accurately handled. These loss mechanisms include, in addition to geometrical spreading loss, bottom reflections, volume absorption, and scattering.

Various methods of handling acoustic wave propagation problems exist, because no individual method adequately deals with the many types of problems and conditions that one is likely to encounter in a variable ocean environment. These methods include ray theory, fastfield theory, normal mode analysis, and the parabolic equation method. An important test of any particular method is its ability to handle substantial variations in environmental parameters, and to accomplish this while using only a reasonable amount of computer time.

The acoustic wave equation cannot be separated if the environment varies in both depth and range, so that direct numerical

integration is required. Since the basic equation represents a three-dimensional boundary value problem, there are currently no practical methods available to solve the problem in this way. Directly integrating the acoustic wave equation, with appropriate boundary conditions, remains impractical due to the excessive computation times, even taking into account the recent leaps forward in computer technology. Thus some simplifying assumptions are always introduced before the wave equation is solved. The parabolic equation (PE) method is thought to represent a viable alternative, as it is an approximation to the wave equation which lends itself to practical numerical solution.

II. Range Independence in the Parabolic Equation Method

Study of the parabolic equation (PE) method was initiated by Tappert and Hardin [42], in an attempt to deal with an ocean environment in which the sound speed, and hence the index of refraction, is dependent upon range as well as upon depth, and possibly azimuth. Since 1974 the PE method has become the standard way of dealing with this "range-dependent" case. However, this is theoretically incorrect, as a basic assumption necessary to the derivation of the PE actually implies that the index of refraction is independent of range. To see why this is so, it is necessary to examine the derivation of the PE in detail. A convenient reference in this regard is [38], and we begin with equation (8) of this paper.

$$\frac{\partial^2 u}{\partial r^2} + 2ik_0 \frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} + k_0^2(n^2-1)u=0. \qquad [38.8]$$

Here, r, z, and θ are the range, depth, and azimuthal variables respectively, $u=u(r,z,\theta)$ is the (unknown) modulation of the spatially

varying part of the acoustic pressure, $c=c(r,z,\theta)$ is the actual sound speed, c_0 is a reference sound speed, $n=n(r,z,\theta)=c_0/c$ is the index of refraction, ω is the acoustic frequency in rad/sec, and $k_0=\omega/c_0$ is the reference wave number. As usual, the farfield assumption, $k_0 r \gg 1$, is made.

Equation [38.8] may also be expressed in operator format:

$$\left(\frac{\partial^2}{\partial r^2}+2ik_0\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+k_0^2(n^2-1)\right)u=0. \qquad [38.11]$$

Two seemingly different methods of deriving the fundamental PE have appeared in the literature, one employing [38.8] and the other [38.11]. To derive the PE from [38.8], the "paraxial approximation," $u_{rr}\approx 0$, is made. One supposed justification for this is the assumption that u varies "slowly as a function of r on a wavelength scale, i.e., $|u_r| \ll |k_0 u|$, which implies that $|u_{rr}| \ll |2ik_0 u_r|$." [29, p.475]. This implication is clearly incorrect, yet it is the foundation upon which the paraxial approximation is based.

The second standard method of obtaining the PE is to replace [38.11] with:

$$\left(\frac{\partial}{\partial r}+ik_0-ik_0Q\right)\left(\frac{\partial}{\partial r}+ik_0+ik_0Q\right)u=0, \qquad [38.12]$$

where the operator Q is defined by

$$Q=\left(1+(n^2-1)+\frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}+\frac{1}{k_0^2 r^2}\frac{\partial^2}{\partial\theta^2}\right)^{1/2}. \qquad [38.13]$$

The difference between the operators in [38.11] and [38.12] is the commutator $C=D_r Q-QD_r$, where D_r indicates the operator of partial differentiation with respect to r. Thus, in order to obtain [38.12] from [38.11], it is necessary to assume that C vanishes (i.e., is negligibly small). After this is done, the second factor in [38.12], which represents the incoming wave, is neglected, yielding

$$\frac{\partial u}{\partial r} + ik_0 u = ik_0 Qu. \qquad [38.14]$$

When the square root operator Q in [38.14] is replaced by the linear terms of its Taylor series, the resulting equation is identical to that obtained by applying the paraxial approximation to [38.8], namely the standard three-dimensional (3D) PE,

$$\frac{\partial u}{\partial r} = \frac{ik_0}{2}[n^2(r,z,\theta)-1]u + \frac{i}{2k_0}\frac{\partial^2 u}{\partial z^2} + \frac{i}{2k_0 r^2}\frac{\partial^2 u}{\partial \theta^2} \quad . \qquad [38.9]$$

To see that the two basic assumptions employed as alternative means of deriving the PE, namely the paraxial approximation and the vanishing of the commutator C, are in fact the same, assuming that the incoming wave is ignored, begin with the equivalent equations [38.8] and [38.11]. Then:

$$u_{rr} + \cdots = 0 \Leftrightarrow \left(\frac{\partial}{\partial r} + ik_0 - ik_0\left[1 + \tfrac{1}{2}(n^2-1) + \frac{1}{2k_0^2}\frac{\partial^2}{\partial z^2} + \frac{1}{2k_0 r^2}\frac{\partial^2}{\partial \theta^2}\right]\right)u = 0$$

$$\Leftrightarrow \left(\frac{\partial}{\partial r} + ik_0 - ik_0 Q\right)u = 0 \Leftrightarrow \left(\frac{\partial}{\partial r} + ik_0 - ik_0 Q\right)\left(\frac{\partial}{\partial r} + ik_0 + ik_0 Q\right)u = 0$$

$$\Leftrightarrow Cu = 0.$$

Therefore, it is necessary to assume that $C=0$ in order to derive the PE. However, in the azimuthally independent two-dimensional (2D) case the vanishing of the commutator actually implies that n is range-independent, while in the 3D case the assumption means that n is "very nearly" range-independent. Thus, in many cases, the basic theoretical reason for the PE method is defeated when the commutator is assumed to be negligibly small.

The computation required to understand this is straightforward. First, observe that the commutivity of D with Q implies that of D with Q^2, since $DQ^2 = DQQ = QDQ = QQD = Q^2D$. Hence, the (approximate) vanishing of $D_r Q - QD_r$ implies that of $D_r Q^2 - Q^2 D_r$. However,

$$[D_r Q^2 - Q^2 D_r]\, u = [D_r(1+X+Y) - (1+X+Y)D_r]\, u$$

$$= 2(nn_r - k_o^{-2} r^{-3} D_\theta^2)u, \text{ where}$$

$$X = n^2 - 1 + k_o^{-2} D_z^2, \quad Y = (k_o r)^{-2} D_\theta^2.$$

In the 2D case $D_\theta = 0$, so $Cu = 0$ implies that $n_r = 0$, i.e., the refractive index is range-independent. In the 3D case, $Cu = 0$ means that $n_r = D_\theta^2(u)/nk_o^2 r^3$. Combined with the farfield assumption, it is clear that $n_r \approx 0$.

It is appropriate to observe that, in their discussion of the approximation resulting from neglecting the commutator, Seigmann, et al might have reached the same conclusion. A basic consideration at this point in [38] is that the term $k_o^{-1} (n^2-1)^{-1} n_r$ must be negligible. Since $n \approx 1$ (and in fact $n = 1$ at various points), this requires that $n_r \approx 0$. The primary error occurs in equation (34) of [38], where "max" should be replaced by "min," and $\min |n^2 - 1| = 0$.

Thus, it is necessary to consider the PE method, from a theoretical viewpoint, as an alternative way of solving the range-independent case, where it is known to perform efficiently (see [6], for example), as opposed to a method of solving the range-dependent case. For the purpose of practical numerical computation, the range-dependent case has been attacked by approximating n with a function that is piecewise constant in r. Numerical schemes employing the PE method, such as the wide-angle split-step Fourier method [43] and the wide-angle implicit finite difference (IFD) scheme [5], incorporate

this idea.

III. The Standard PE Method, and Alternatives

There are several basic assumptions which must be made in order to apply the standard PE method. These include: the far-field approximation; the index of refraction is piecewise constant; there is only an outgoing wave; there is no scattering; the frequency is less than 500 Hz; the propagation angle is small (less than about 15°). One way of possibly avoiding some of these assumptions is to use a different method of solving the acoustic wave equation.

A careful analysis of the errors inherent in the standard PE method, for the range-independent case, is carried out by Fitzgerald [16]. The importance of choosing the reference sound speed, c_0, to be close to the average value of the phase velocities c_m for excited modes, is quantified. The PE errors are smaller for narrower bands in phase velocities or, using the ray mode analogy, the narrower the cone of propagating rays. Also, it is shown that the PE method maintains a given accuracy to longer ranges the lower the frequency. Since computation time increases dramatically with increasing frequency in the PE method, this indicates that as low a frequency as practical should be employed. Also, Fitzgerald derives a compact expression for the maximum range of effectiveness of the PE method. When this is applied to a frequency of 100 Hz, in a range-independent environment, with a simple surface and bottom, he estimates a maximum effective range of 11 km. For a 10 Hz source located near the surface, so that the RSR modes are the only excited propagating modes, this increases dramatically to 15000 km. Overall, Fitzgerald shows that the PE

method properly accounts for mode coupling when backscattering is negligible, and properly accounts for mode propagation when there is a small spread in mode wavenumbers. He suggests enhanced PE methods which might work when these conditions fail, but does not discuss the greatly increased computation time that would likely be required to carry out these interesting suggestions.

A different approach is considered by DeSanto [13], who relates the solutions of the general Helmholtz equation and the PE via an integral transform, which is exact for range-independent sound speeds. However, there is no estimate given for the error introduced when the Helmholtz equation is replaced by the PE. It appears that DeSanto's interesting ideas could lead to such estimates, which would be important if they could be obtained.

An alternative approach to the numerical computation of the wide-angle acoustic field in the range-dependent case is given in [15] by Estes and Fain. They also proceed stepwise, by employing a fully defined field in one vertical plane to develop the field at a plane a fixed distance down range. The propagation is composed of two parts, somewhat analagous to the predictor/corrector methods in differential equations: propagation through a homogeneous space, and a correction due to the fact that the environment is not homogeneous. They approximate the square root operator by the terms of its Taylor series up to order 4, but the results have not been coded, so numerical evidence is unavailable. Although their method seems to avoid the phase errors that are an inherent part of the standard (narrow-angle) PE approach, and thus offers an improvement over that method, it

appears that the recent development of the wide-angle PE supercedes the results of Estes and Fain. Another attempt, the first order continued fraction approximation of Berkhout [4], is equivalent to the standard narrow angle approximation, and his second order continued fraction result is just the Claerbout (wide-angle) approximation.

Fitzgerald [16] discusses the possibility of employing normal mode theory to handle the range-dependent case, by dividing the range into intervals within each of which the sound speed is assumed to be range-independent. Note that, in view of the range-independence of the PE method described above, this technique is also required (and is used extensively) when applying the PE method to the range-dependent case. This piecewise constant normal mode approach is not analyzed by Fitzgerald. Other possible approaches to solving the acoustic wave equation will be discussed below.

IV. Solution Methods for the Parabolic Equation

The original numerical method of solving the PE is known as the split-step Fourier method. The basic equation of the split-step Fourier method is an exact solution to the PE only for constant index of refraction n. Since vertical sound speed changes are usually less than 5%, and strong near-surface horizontal sound-speed gradients are roughly 0.1 m/s/km, the split-step method will often give a valid solution to the wave equation for a real environment.

The split-step method of solving the PE is fast and accurate when bottom effects are weak, but less efficient when bottom interaction is strong. A viable alternative is the implicit finite difference (IFD) method, which is usually comparatively fast, and

unconditionally stable. IFD is applicable when, as the PE solution is marched out in range, the boundary information at the advanced range step can be expressed in terms of the known values at the present range step. The Lee-Botseas IFD model [25] is applicable to both range-dependent and range-independent environments, and can handle arbitrary surface and bottom boundary conditions, horizontal interfaces of layered media (with densities constant on each side of the interface; see [29] and [31] for details of the treatment of horizontal interfaces), attenuation, shallow or deep water, and shallow to deep or deep to shallow propagation. As a matter of necessity (as explained above), in a range-dependent environment the SVP can be updated only abruptly at the range steps, and cannot change within a step. Advantages of finite-difference methods over split-step methods, in some idealized range-independent cases, are discussed by McDaniel [30]. A complete analysis of the well-posedness of the IFD scheme for the solution of the wide-angle PE is given by St. Mary and Lee [40].

The split-step algorithm solves a pure initial-value problem, by imposing an artificial zero bottom boundary condition and pressure-release surface condition. The implicit finite difference (IFD) scheme, on the other hand, is designed to treat the bottom-boundary condition exactly. Furthermore, since strong bottom interaction is often associated with a wide propagation angle, the wide-angle IFD scheme [9] could prove to be especially useful in handling cases where both of these phenomena occur.

The method of lines has also been employed to solve the PE.

This technique is described by Lee and Papadakis, who call it the ODE method. One advantage of this approach is that, if the bottom boundary conditions are known, no subbottom knowledge is required. The ODE technique needs considerable computer memory capacity, but if this is available the method is efficient. The computer memory requirement, in fact, can be reduced by applying the Generalized Adams Method (GAM) described by Lee, Jackson and Preiser [27]. They claim that GAM reduces computation speed tremendously as well. McDaniel, Saad and Lee [32] also discuss the GAM.

V. Bottom Conditions

The broad effect of bottom loss on ocean sound propagation is an increasing loss with decreasing frequency. In actuality, the bottom is constituted of many sedimentary layers, each acting acoustically like a fluid. The sound speed in the bottom is usually close to that of the water, but the attenuation is much greater in the bottom. We describe some of the ways which have been devised to treat the various types of bottom conditions which can occur.

One method of representing the bottom, chosen by Guthrie and Gordon [18], is as a series of fluid layers, where in the ith layer the sound speed varies as

$$c(z)=c_i[1-2g_i(z-z_i)/c_i] \ ,$$

where c_i and g_i are the sound speed and sound speed gradient at the top of the ith layer, and z_i is the layer depth. Their specific example has 5 layers, with the fifth one having a small negative gradient. This is necessary to satisfy the mechanics of the normal mode starter program of the PE method, and has no effect on the

solution, as its depth is below that to which significant acoustic energy penetrates. A variable jump discontinuity in the sound speed, of 11% on the slope and tapered to zero on reaching the ocean basins, is applied at the water-sediment interface. This allows for greater bottom reflectivity on the continental slope as compared with the ocean basins, and compensates for the inability to include density in the model. Finally, they choose a constant attenuation coefficient in the bottom of 0.1f dB/km (f=frequency in Hz), corresponding to a porosity of at least 65%.

It is obvious from this description that Guthrie and Gordon carried out a very detailed analysis of the bottom structure and properties before applying the PE method. As the PE method is quite sensitive to errors in the knowledge of the bottom environment, a similar bottom analysis will be required for any application of the PE method to propagation loss calculations to succeed.

When the split-step method is used to solve the PE, the bottom conditions must be modeled in such a way that the potential function goes smoothly to zero at maximum depth. One way to accomplish this is to extend the bottom so deep that the sound propagating downwards does not reach the lower boundary within the horizontal range being considered. This is not practical, however, because of the large increase in the number of necessary FFT points that results. Another possibility is to introduce a sufficiently high bottom absorption, but it can't be arbitrarily large, since it must represent the actual attenuation. Mathematically this is accomplished by incorporating a volume absorption coefficient, which increases rapidly with depth

until the acoustic field becomes negligible, into the PE. In practice a combination of these two methods can be employed, with the depth extension and the absorption chosen to fit the particular environment being considered. Another method of accounting for bottom absorption is to add a small imaginary part to the wave number in the bottom.

In many ocean environments differences in temperature, pressure and salinity, as well as periodic deposits of sedimentary material, create a layered medium in both the water and the bottom. Various attempts have been made to adapt the PE method to treat the interfaces between such layers. In several papers [28, 29, 31], McDaniel and Lee use finite difference techniques to handle such boundary conditions. One drawback of this work is that the mesh spacing is uniform, introducing computational difficulty when there are mesh points which do not lie exactly on the interface boundary. This problem was addressed by deG Gribble [12], who employed a variable mesh spacing. An implicit finite difference (IFD) computational technique, incorporating the horizontal interface ideas of McDaniel and Lee and the wide angle equation of Claerbout [11], is currently undergoing extensive testing. It remains unclear how satisfactory these methods are, since substantial experimental evidence is lacking.

Although the PE method does not take the density change at the water-sediment interface into consideration, so that a somewhat greater bottom reflection loss than normal is to be expected, this is unlikely to be a source of serious error. For example, a density contrast of 1.25:1 causes a decrease in reflection loss at low grazing

angles of less than 0.5 dB/bounce, while a 5% jump in sound speed at the sediment interface causes a more meaningful 4 to 5 dB decrease [18].

VI. Ice and Other Environmental Concerns

The importance of accurate knowledge of environmental conditions, when doing propagation loss calculations, is abundantly clear in the results described by Ellis and Chapman [14] for low-frequency shallow water propagation. In one experiment, curves of propagation loss versus frequency at a range of 55 km are given for two sites; one with a chalk bottom and the other a nearby site with a sand bottom. For frequencies less than 100 Hz, the loss observed over the chalk bottom was more than 15 dB greater than that over the sand bottom! Thus, certainly for shallow water low-frequency propagation, knowledge of the bottom conditions is absolutely essential if accurate calculations are to be made.

A triangular ridge model has been shown by Greene and Stokes [17] to give better agreement with backscatter data from Arctic sea ice than a standard Gaussian model. Their model and data confirm the observations of Mellen and Marsh [33] and Brown and Milne [7, 8, 34], that such backscatter levels are much stronger than those in the open ocean. This is caused by the large linear ridge-keel structures which interrupt the otherwise smooth stretches of ice on the underside surface of Arctic sea ice. Furthermore, small-scale roughness, due to the rubble which makes up the keels, is superimposed on the linear structure.

Since a fundamental premise in the PE method is that there is no backscattering, it is clear from the above that propagation loss calculations employing the PE method will likely contain very large errors in an Arctic underice environment. One possible mitigating factor could be the use of low frequencies, as the local backscattering strength of a patch on a ridge is, according to Bass and Fuks [2, p. 108], proportional to the fourth power of the acoustic frequency. Since most of the published data involves frequencies greater than 1 kHz (Greene and Stokes [17], for example, exhibit plots of backscattering strength at 1.81 kHz), the possibility remains that the PE method will give reasonably accurate results in the 100 Hz range. Another factor to consider in this regard is that PE approximations are better at low grazing angles, which is precisely the case for which the backscattering probability is greatest. To more fully answer these questions, it will be necessary to carry out detailed and precise Arctic experiments, similar to those undertaken in the South Tasman Sea in 1975 and described by Guthrie and Gordon [18].

The knowledge of bottom properties and gross sound-speed characteristics alone will probably not be sufficient to obtain an accurate model of acoustic propagation under ice. For example, there may be energy loss as a result of mode coupling, either from ice-bottom roughness where there is no deep sound channel, or from inhomogeneities within the water column in the boundary region between different water mass types. The latter could be caused, for example, by marked changes in salinity as the sound propagates past partially

melted ice flows.

Since rays which propagate at steep angles are greatly attenuated by radiation and bottom absorption, low frequency long range propagation is dominated by rays having small grazing angles. In this environment, therefore, the standard PE is usually acceptable. However, when the reference wave number differs appreciably from the wave numbers of the propagating modes, phase errors are introduced in the standard PE. These errors are not reduced by reducing the range step or increasing the number of FFT points in the split-step solution of the PE. A major advantage of the wide angle PE is that these phase errors are essentially eliminated.

The PE method has been applied to wide angle propagation by employing a more accurate, rational approximation to the square root operator Q defined above. Claerbout [11] introduced this means of attack, by using $(1+3x/4)/(1+x/4)$ to approximate $(1+x)^{1/2}$, in place of the estimate $1+x/2$ which results in the standard narrow angle PE. In this context, "narrow angle" means less than about 15°, while "wide angle" is less than about 40°. A "very wide angle" (less than about 70°) PE, which utilizes a second order rational approximation for the square root operator, also exists, but its utility has not been demonstrated.

Boundary fluctuations, source directionality and water column variations are three mechanisms which can introduce azimuthal variation in the sound pressure field. The development of three-dimensional (3D) capability in the various PE models has only recently been begun, and the results of realistic test examples are not

available yet. Siegmann and Lee [39] offer a more detailed explanation of why such techniques are important. They also develop estimates of ranges for which a 3D PE would be valid. For example, their estimates mean that at 150 Hz the range must be at least 40 m, while it must be at least 400 m when the frequency is 15 Hz.

VII. Choice of Parameters

When studying ducted propagation in the ocean, there is generally an optimum frequency to consider. Such a phenomenon occurs because of competing attenuation and propagation mechanisms at various frequencies. At high frequencies, scattering and volume loss simply increase with frequency, but at low frequencies the situation is more complex. Here the intensity of sound within the duct is affected by propagation and attenuation mechanisms in the bottom and otherwise outside the duct. As sound penetration into the bottom increases with decreasing frequency, the overall attenuation of sound throughout the water column will increase as the frequency decreases. Therefore both high and low frequencies can result in high attenuation, with intermediate frequencies having the lowest attenuation. For example, an optimum frequency on the order of hundreds of hertz occurs in a typical shallow water environment.

The importance of the choice of receiver depth is made clear in the results of Jensen [20]. He discusses a simple example of propagation in shallow (110 m) isothermal water, modeled by normal mode theory, at a frequency of 100 Hz. In this case the theory and experimental data agree extremely well. Both show a propagation loss level that is 20-25 dB greater for a receiver depth of 1 m than that

for a receiver depth of 50 m, over the entire range of 30 km.

The ideal vertical step size in applying the split-step Fourier algorithm to numerically solve the PE is $0.2\lambda\sin\theta$, where λ is the wavelength and θ is the grazing angle. This places an upper limit on the number of modes, and hence the frequency, that can be handled, since computer run-time is proportional to n log(n), where n is the number of FFT points. A deterministic criteria for the ideal horizontal step size, Δr, does not appear to be known, although it is certainly limited by the frequency, and the magnitude of horizontal gradients. In practice a trial and error approach is usually taken, where Δr is decreased until the solution converges.

It is essential to the PE method that the initial pressure distribution over depth includes full phase information (coherent addition of modes). Three methods of starting the PE model are normal mode theory, ray theory, and assuming a Gaussian-shaped initial pressure distribution near the source. The latter is appropriate for deep water, because it filters out high-angle energy (which would be lost anyway) and prevents aliasing of this energy into low angles during the FFT sampling process. Initializing the PE model by normal mode theory requires a constant water depth for the interval in which the mode theory is being carried out, and imposes an upper limit of about 60 degrees on the equivalent ray angle. The ray starter seems best for propagation down the continental slope and into deep water. In this case very high angle energy close to the source must be considered, because of the considerable reduction in the angle during propagation down the slope. On the basis of the requirement that

energy which insonifies convergence zones when it reaches deep water must be treated properly at the top of the slope, Guthrie and Gordon [18] choose a vertical sample interval of $\Delta z=2H/5m$, where H is the water depth and m is the highest order mode required in deep water. Since at 100 Hz it is necessary to consider about 150 modes in deep water, a step size of 12 m is called for in 4500 m water. At the top of the continental slope, where H is about 350 m, a step size of about 1 m is adequate.

In the extensive comparisons of experimental data with PE model predictions described by Guthrie and Gordon [18], at frequencies of 63 and 125 Hz, the range step size Δr varied from 10 m on the continental shelf to 200 m in deep water, while the depth step size Δz varied from 6 m to 12 m, with the greater Δz also taken in deep water. However, they speculated that under certain circumstances a Δz of 3 m or less would have improved the results, but such an option was not available to them in their model. This indicates that the choice of vertical step size must be made with considerable care. Δz must be small enough to ensure that the highest order modes needed are sampled properly, but not so small that computer run-time becomes excessive. In the interest of efficiency, Δz should be chosen automatically by the model to suit local conditions.

Jensen and Krol [19] carry out a detailed analysis of the allowable range step, Δr, when the split-step Fourier method is used to solve the PE. They conclude that for deep water Δr can be taken as large as 100 or even 1000 m. In general, it is required to take Δr inversely proportional to both the maximum vertical sound speed

gradient and to the frequency. Furthermore, Δr is slightly dependent upon bottom attenuation and source/receiver depths. They recommend, as did Guthrie and Gordon, that Δr for a given propagation problem be determined empirically, by reducing it until a stable solution has been obtained. Finally, Jensen and Krol observe that in a shallow water environment it appears that Δr must be extremely small.

VIII. Comparisons With Other Models and Experimental Data

In order to determine the accuracy and applicability of the PE method, it must be compared with both experimental data and with the predictions resulting from other models. While agreement with other models instills confidence in any particular method, the final check on an acoustic model is a comparison with the results of realistic experiments. Only then should we be convinced that the model being evaluated includes all of the necessary and correct physics and mathematics for understanding and explaining sound propagation in a real ocean environment.

An example of a complicated North Atlantic environment modeled by adiabatic normal mode theory is presented by Jensen [20]. The water depth varies between 115 and 305 m, with a maximum bottom slope of about 1°. The 65 km range is divided into two distinctly different water masses, with two different SVP's. The bottom is also of two types, sand and silt, with the changeover occurring at a range different from that where the SVP changes. In spite of the difficult environment, Jensen found excellent agreement between theory and experiment. Over almost 6 octaves (50-2540 Hz) of frequency, and over the entire range of 65 km, the maximum deviation was only a few dBs.

If the calculation times (which are not mentioned) required to obtain these results are at all reasonable, this strongly suggests that adiabatic normal mode theory is a very viable alternative to the PE method for a range-dependent environment. It should be emphasized, however, that in this example an extremely detailed environmental description played a crucial role in the achieved accuracy of the modelling results.

A detailed experiment of propagation over a seamount is described by Jensen ([20],[21]), with the PE method used as a model. Although he calls this a "strongly range-dependent environment (14° mean slope across the mount)," the SVP remains constant over the entire 140 km range. Thus the common terminology would refer to this as a range-independent case. At any rate, the agreement between theory and experiment is uniformly good over the full 140 km range, for frequencies ranging from 12.5 to 400 Hz. Moreover, in this case the agreement is achieved without very detailed environmental information. This indicates that the real SVP is well approximated by the constant one that is assumed, and that the important physics is associated with reflection off the seamount and with propagation in the water column. Thus these results can be somewhat encouraging for the study of under-ice propagation, although backscattering off ice keels and SVP changes due to salinity differences are still likely to present significant difficulties, as described earlier.

Comparisons of normal mode, fast field program (FFP), and PE wave theory models for test cases involving four different underwater acoustic environments are described in [22] by Jensen and Kuperman. A

split-step algorithm (PAREQ) is used to solve the PE. The first case involves a range-dependent environment, where the SVP remains constant for the first 20 km, transitions for 10 km, and remains constant again for the final 20 km. The frequency is 25 Hz. For receiver depths of 25, 250, and 400 m the three models agree quite well over the first 20 km. The normal mode solution is stopped at this point, because of the range-dependent SVP. However, when the FFP and PE solutions are continued out to the maximum range of 50 km, they diverge substantially from each other. Jensen and Kuperman argue for accepting the PE solution, stating that "the PE gives the correct result in the range-independent part of the environment and because of the mathematical nature of the PE solution there is no reason to believe that the accuracy of this solution diminishes in the mildly range-dependent region." We question this argument, because of the proof given earlier that the derivation of the PE implies a range-independent SVP.

The second case in [22] is a range-independent environment with different bottom speeds. The frequency is again 25 Hz. As the maximum propagation angle increases with increasing bottom speed, and as the standard PE is valid only for narrow angles of propagation, the agreement between the PE solution on the one hand, and the apparently accurate FFP and normal mode solutions on the other hand, deteriorates with increasing bottom speed. This offers a good illustration of the need for the wide angle approximation to the PE, discussed elsewhere in this paper.

Case 3 in [22] is a range-independent shallow water (100 m)

environment, at a frequency of 250 Hz. With the source and receiver depths both 50 m, the agreement between the three models is excellent over the full range of 10 km, except for 3 unexplained isolated spikes in the PE curve, which overpredict the propagation loss by as much as 35 dB. When the source and receiver are located just off the bottom, so that higher order modes are excited, the PE solution no longer agrees as well with the correct normal mode and FFP solutions. The reason for this is the increased inaccuracy in the reference sound speed when more modes are excited, as we discuss elsewhere.

The fourth case in [22] deals with a fairly complicated range-independent environment, where the bottom is sloping and has a sediment layer. The frequency is 25 Hz. Agreement between the three models is quite good, even out to a range of 150 km.

In addition to the results of the four test cases just described, Jensen and Kuperman [22] list the basic numerical parameters and associated computer times for the normal mode (NM) and PE solutions. The depth step Δz varies from 0.5 m (case 3) to 8.5 m (case 4) for NM, and from 0.26 m (case 3) to 11.2 m (case 2) for PE. The range step Δr varies from 10 m (case 3) to 500 m (case 4) for NM, and from 2 m (case 3) to 100 m (cases 1 and 4) for PE. The number of FFT points for the PE is 256 for case 2, and 1024 for cases 1, 3 and 4. The calculation time on a UNIVAC 1100/60 varies from less than 1 min (cases 2 and 3) to 2 min (case 4) for NM, and from 1 min (case 2) to 30 min (case 3) for PE. The PE method is particularly slow when there is heavy bottom interaction (30 min in case 3 and 15 min in case 4).

Examples of the application of the IFD technique to several problems are described by Lee and Botseas [25], and compared to solutions obtained by normal mode theory, and by Jensen and Kuperman [22] using the split-step method of solving the PE. In general the solutions agree quite well. The reader is referred to [25] for the details.

Guthrie and Gordon [18] describe extensive propagation loss calculations and experiments carried out in the south Tasman Sea in 1975. This appears to be the most definitive study to date on the appropriateness of the PE method in a realistic ocean environment. The computer runs involve accurate bathymetric and sound-speed data for ranges up to 2500 km at frequencies of 63 and 125 Hz. Among the interesting and important results obtained were:

1. Initialization of the PE model in a shallow, coastal environment was accomplished by a ray trace and a normal mode prediction, and in both cases the models predicted absolute levels in deep water that were within 5 dB of those obtained experimentally.

2. The optimum vertical sample size was difficult to determine when running a PE prediction from shallow to deep water, or vice versa.

3. The assumed properties of the sediment cover had a marked effect upon the predicted acoustic shadow of substantial underwater features.

4. The model failed to predict an observed increase in shadow depth with frequency.

5. The clarity of the convergence zone structure varied

strongly with range, in certain range-dependent environments.

6. The predicted acoustics were very strongly dependent upon the SVP, with even subtle changes in sound-speed profiles having a marked effect.

7. At very low frequencies (16 and 31.5 Hz), the relatively weak source level and high ambient noise conditions resulted in poor quality data.

Several other detailed comparisons of experimental data and PE predictions are given by Guthrie and Gordon [18]. The reader is referred to the original paper (which should be required reading for anyone in the field) for the precise results. We only observe here that, although agreements between experiment and prediction are usually reasonably good, discrepancies on the order of 10 dB or more are quite common. Moreover, on occasion the errors were so large as to render the particular PE model being employed useless, and requiring ad hoc adjustments to the model (such as changes in the bottom assumptions) to bring the predictions back into line with the data. This clearly indicates the importance of achieving accurate knowledge of the propagation characteristics of the region in which one wants to employ the PE method, before the actual work is carried out.

Jensen and Krol [19] give propagation loss predictions for several cases, employing the PE split-step model, a normal mode model, and a ray-tracing model. For a deep water (5000 m) range-independent case, with a frequency of 100 Hz and a range of 65 km, the curves generated by the three models agree very well. The range step Δr for

the PE program is 1000 m. On a UNIVAC 1106, the computation times were 6 min for the ray program, 12 min for the mode program, and 24 min for the PE program. For a deep water range-dependent case, with a frequency of 250 Hz and a range of 32 km, ray-tracing and PE predictions are compared with experimental data, and in both cases agreement is quite good. Δr for the PE program is 100 m, and calculation times are 4 hours for the ray-tracing program and 28 min for the PE program. For a shallow water range-independent case, the results of a normal mode model and the PE model agree very well. However, a Δr of 2 m is required to obtain a stable solution for the PE model. This small range step results in a calculation time of 3.5 hours for the PE program, whereas the mode program takes less than one min to execute.

The results of several numerical computations are shown in [1]. They all deal with the case where the SVP is range-independent, and the normal mode solution is used as a reference. For a narrow angle (5.5°) test case, agreement between the normal mode solution and the PE solution employing the IFD scheme with interface is quite good. The phase error in the standard PE is evident in a wider angle (19°) test case, but the wide angle PE gives excellent agreement with the normal mode reference solution. For a very wide propagation angle (70°), no available PE method produces a satisfactory result.

Bates [3] implemented a range-independent, boundary-free example with an SVP chosen so that the wave numbers and modal depth functions could be obtained in closed form. The split-step algorithm for solving the standard PE was then tested, by comparison with the

closed form solution, for various choices of frequencies, number of FFT points, and sound speed gradients. To indicate the frequency dependence, for a single propagating mode at 50 Hz, a range increment of 400 meters and 64 FFT points was sufficient to generate results within 1 dB of the closed form solution. When the frequency was increased to 500 Hz (and all other parameters remained the same), the error increased to 5 dB, and at 1 kHz it was 11 dB. When the number of FFT points was varied, with the frequency held constant at 1 kHz and 1 propagating mode, a doubling in the number of FFT points resulted in a reduction in the error by a factor of 4. For example, the error was 42 dB with 32 FFT points, 11 dB with 64 FFT points, 2.75 dB with 128 FFT points, and .67 dB with 256 FFT points. As expected, an increase in the sound speed gradient resulted in an increase in the error, and an increase in the number of propagating modes from 1 to 3 caused the introduction of phase errors. It is interesting to note that the magnitude of these phase errors was independent of the number of FFT points, at least for the limited number of examples considered by Bates. Also interesting is the fact that all of the errors discussed here were essentially independent of range, for those ranges considered (up to 10 km).

Results of a test of the wide-angle IFD model are reported in [5]. The test case is a range-independent environment, with an isovelocity water column over an isovelocity half-space bottom. The frequency is 250 Hz, the water depth is 100 m, the maximum range is 10 km, and the source and receiver are just off the bottom. Because of this last condition the higher modes are more strongly excited, and

all 11 modes of this example are required for an accurate solution. The reference solution is one gotten from the fast-field program (FFP), which has been compared with normal mode results and is believed to be correct. In this test the wide-angle IFD results are in excellent agreement with the FFP reference solution. When the narrow-angle (standard) IFD scheme is used, the expected phase errors appear.

Comparisons of experimental propagation loss data with predictions made by the PE method and by ray tracing are described in [6], for convergence zone propagation over 60 km, at a frequency of 550 Hz, with a range-independent SVP. The PE solution agreed with the measured data to within 1 dB, and the ray trace solution agreed within 1.5 dB. Note that in this case, of the first convergence zone, there is very little bottom interaction, so that the bottom handling capabilities of the PE method were not tested in this experiment. Furthermore, when multiple SVP's measured along the propagation track were input to the PE, and the resulting predicted propagation loss was compared with experimental data, the comparison was not significantly better than that obtained previously by employing a single SVP over the entire track. Thus, incorporating SVP range-dependence did not appreciably improve the results.

Chan et al [9] point out the existence of an unconditionally stable IFD scheme, which discretizes the standard 3D PE by means of central finite differences for both the depth and azimuthal derivatives, and then applies the Crank-Nicolson method. The results appear to be reasonably accurate, but the scheme requires excessive

computer time. In [9] they introduce a stability-generating dissipative term into a basic finite difference scheme for the standard Schrödinger equation, and analyze and test the result. In their test case this explicit scheme, using the same range step (0.001m) as in the Crank-Nicolson implicit scheme, produces essentially the same results, but 3.5 times faster in terms of CPU time. The actual computation times are not mentioned. The accuracy of the computed solutions is measured by comparing them to what is claimed to be an exact solution, $u=\sin(\alpha z)\exp(im\theta)\exp(im^2/(2k_0 r))$. However, it is straightforward to show that if this function u satisfies the standard 3D PE, then $\alpha^2=k_0^2(n^2-1)$. Since α is constant ($\pi/100$ in their test case), this requires that n is also constant, i.e., the range-independent case. In fact, the last equation gives an explicit relationship between α, k_0, and n, which must hold for this test case to be valid. The test frequency is not mentioned in [9], so it is not clear if this consideration was taken into account. The test sound speed, however, was taken to be constant, by necessity (as explained above), not for "simplicity" (as claimed in [9]).

In [10] Chan et al describe, in somewhat more detail than in [9], various finite difference schemes for solving the PE. They define a "practical stability condition," and compare three stable explicit finite difference schemes with the Crank-Nicholson method. The greatly oversimplified equation $u_r=(i/2k_0)u_{zz}$ is used as a test case, with a depth of 100 m, a frequency of 500 Hz, a constant sound speed of 1500 m/s, a range of 200 m, a depth increment of 1 m, and a range step of 0.001 m.

Apparently this small range step was required in order for the four methods to agree with the exact solution to within 1%. In spite of the simple equation being solved, the lack of variation in the environmental parameters, and the short range, the CPU times required were substantial. For the three explicit schemes, these times ranged from more than 48 minutes to almost 1 hour and 22 minutes, while the Crank-Nicholson method took over 2 hours and 7 minutes of CPU time.

IX. Possible Advantages of the Parabolic Equation Method

Although ray theory, particularly at higher frequencies, generally requires much less calculation time than wave models such as the PE method, ray theory is incapable of handling diffraction. This causes a ray theory model to predict zero sound in shadow regions which, although unrealistic, might be satisfactory for the many applications where the primary interest is in sound levels in the insonified regions. In other cases, where diffraction is important, the PE method or some other wave model must be used, because only such models accurately predict the frequency dependence of the diffracted field and hence give correct field levels in the shadow zones.

In a study of propagation in a wedge-shaped ocean with a penetrable bottom [23], Jensen and Kuperman observe that the PE method performs a reasonably good job of modeling the modal cutoff that occurs during upslope propagation. Adiabatic mode theory, on the other hand, fails to give good results in this case, because of the penetrability of the bottom.

In the PE model both the SVP and the bottom depth and structure are allowed to vary in range. Attenuation in the bottom is

included, by introducing a complex sound speed.

Although the effect of ocean current profiles on sound propagation is at least an order of magnitude less that the effect of sound-speed profiles, the cumulative effect of current profiles may cause significant changes in PE predictions of underwater acoustic fields. The effect of current on reciprocity is more pronounced. It is easy to numerically include current profiles in the PE model, with a negligible increase in program complexity or calculation time. In cases where one is interested in absolute levels of propagation loss, the ability to include currents is certainly an important feature. When relative loss levels are of concern currents would probably not matter as much, since they would have the same effect on different signals of the same frequency. However, current considerations might very well play a role in the choice of search depth and pitch angle. Phu and Tappert [35] give a detailed discussion of why and how current should be incorporated in PE models.

The phase errors inherent in the application of the standard narrow-angle PE have been essentially eliminated by development of the wide-angle PE. The key to the derivation of the wide-angle PE is the approximation of the square root operator, which naturally arises when the PE is being formulated, by a nonlinear, rational function, as opposed to the linear approximation which yields the standard PE. A detailed derivation of the wide-angle PE is given by Siegmann, Kriegsmann and Lee [38]. They include an error analysis, which we believe is flawed, as described in this paper. A split-step algorithm for the wide-angle PE appears in the paper of Thomson and Chapman

[43]. They point out that existing computer codes for the split-step solution of the standard PE can be easily modified to incorporate the wide-angle capability, with very little change in computation time.

The PE method allows the acoustic wave equation to be treated as an initial value problem, which is amenable to a marching solution. In contrast the elliptic equation, from which the PE is derived, requires boundary conditions which must be prescribed on a closed surface. Thus the PE method is computationally much more efficient than trying to numerically solve the elliptic acoustic wave equation. Also, it is claimed by Tappert and Hardin [42] that the split-step Fourier method of solving the PE is unconditionally stable, even when the index of refraction n is a function of both range r and depth, and that the method has second order accuracy in r. Thus, theoretically if the range step Δr is sufficiently small, the inherent error in the solution can be made negligible. Finally, Jensen and Krol [19] state that the PE method yields a wave solution that includes all mode coupling and diffraction effects.

The PE method is able to handle sloping bottoms. By partitioning the range into increments in which the SVP is range-independent, and then piecing these range increments together via the split-step or IFD technique, the PE method is able to numerically handle the range-dependent case.

Diffraction and all other full-wave effects, such as the rigorous treatment of caustics, are included in the PE method. Using the standard PE, discrete modes are propagated with the correct amplitudes and mode shapes, but with errors in the phase and group

velocities. These errors are essentially eliminated by the wide-angle PE.

A range-dependent environment may be treated by the PE method, by a ray model, or by the adiabatic approximation to the normal mode model. Of these, the first two are generally more accurate than the third. The computation time required increases with frequency for wave models (mode, fast field, PE), while frequency doesn't effect the time for ray models. The PE model requires essentially the same calculation time for the range-dependent and range-independent cases, while ray and mode models' time consumptions are proportional to the number of profiles.

X. Possible Disadvantages of the Parabolic Equation Method

Excessive computation times required in a shallow water environment have been a major drawback for the PE method. However, Jensen and Kuperman [23] report increases in speed by a factor of 100 when the PE model is run on a dedicated computer system in connection with an array processor, as opposed to a general-purpose computer. This obviously makes the PE a more practical alternative when calculating propagation loss in shallow water.

Computation time for the PE method increases with the square of the frequency, so that it is impractical for higher frequencies. Another disadvantage is that is difficult to handle shear propagation in the bottom. Also, the PE method is thusfar unable to handle scattering. A possible method of handling volume scattering in acoustic propagation, by means of an ordinary differential equations approach to the PE, is offered by McDaniel, Saad and Lee in a recent

paper [32]. However, backscattering must be negligible for application of the PE method.

Jensen and Krol [19] observe that steep sound-speed gradients, such as those that occur at the water/bottom interface in shallow water, cause excessive computer run-times for the PE method. From this, together with the claim that the PE method handles range dependence in a simple way, they conclude that the primary application of the method should be "low-frequency acoustic modeling in deep water (no bottom effects) under range-dependent environmental conditions." Although the application of the PE method to shallow water propagation and the incorporation of bottom effects has been improved since 1975, as described elsewhere in this paper, we question whether the method in fact handles range dependence in a "simple way." Our concerns in this regard are described in detail above.

As pointed out elsewhere in this paper, the small range steps necessary in the PE method when there is a large vertical sound speed gradient, such as in a shallow water environment at the water/bottom interface, result in excessive computation times. Also, as the size of the range increment in a numerical implimentation of the PE method is inversely proportional to the frequency, the method rapidly becomes too time-consuming as the frequency increases. At the current state of the art in coding for the PE method, the upper frequency limit, when the program is run on a computer such as the VAX 11/780, is about 500 Hz. On a super-computer this upper limit could obviously be raised, while on a microcomputer the highest practical frequency would be much less than 500 Hz. However, if an efficient dedicated

microprocessor could be developed, it is conceivable that frequencies as high as 1 kHz could be handled by the PE method without inordinate difficulty.

A starter solution, obtained from normal mode theory or some other method, is required in order to initialize the PE method. This can significantly increase the computation time required. Also, the reflection that can occur due to a range-dependent environment cannot be accounted for by the PE method, since all energy is assumed to propagate in the forward direction.

The direct modeling of boundaries is not allowed for in the PE method. Surface modeling is accomplished by making the modal depth function reflected about the surface with a 180° phase change. Bottom absorption is accounted for by modeling the bottom as a change in sound speed with an imaginary part.

Ellis and Chapman, in a recent and very interesting paper [14], describe and combine all extensions to the Pekeris model, which has proven to be a very useful tool in the study of low-frequency propagation in shallow water. This simple model contains all the basics of normal mode theory, and does not deal with the PE. Nevertheless their results are of interest for the current study, both as a means of comparison with PE results, and for the informative light which they shed on shallow water, low-frequency propagation. In particular, the model accurately predicts the very high losses at frequencies less than 200 Hz resulting from shear waves in a chalk bottom. The reason that the modified Pekeris model performs much better in this environment than the usual normal mode models appears

to be that the shear wave speed in the bottom is roughly of the same order of magnitude as the speed of sound in water, whereas other normal mode models require that the shear wave speed be either much less or much greater than the water sound speed. Whatever the reason, the modified Pekeris model gives much better results, under these environmental conditions, than any model based upon the PE that we have seen. Hence, it should be considered as a very desirable alternative to the PE method for low-frequency shallow water propagation.

XI. Recommendations for Future Work

1. Development of a dedicated microprocessor, utilizing the PE method to solve the wave equation.

2. Determination of the speed and accuracy of the Pesogen (Parabolic Equation Solution Generator) system, designed for shipboard use by Tappert, Phu and Daubin. Information on this system seems to be unavailable in the open literature.

3. Estimate the errors inherent in the PE, and in the other adaptations of the acoustic wave equation.

4. Perform a sensitivity analysis, to determine the effect of errors in the input parameters upon the solution of the PE.

5. Develop a method capable of handling a rough surface, such as will be encountered in an underice environment.

6. Develop a method capable of handling scattering, such as will be encountered in an underice environment.

7. Develop a very wide angle solution.

8. Develop a broadband capability.

9. Improve the capabilities of PE models to handle three-dimensional problems.

10. Incorporate the consideration of fronts and eddies in the PE models.

11. Develop methods of more accurately measuring the sound speed profile in an underice environment.

12. Develop more accurate methods of modeling and handling surface and bottom boundary conditions.

13. Develop automatic step-size determination.

14. Develop the ability to handle shear waves.

15. Include multiple irregular interfaces in a practical model.

XII. Conclusions

With the current state of the art in propagation loss modeling, it appears hopeless to expect accurate absolute measures of loss levels except in the very simplest of environments. However, it is quite likely that reasonably good comparative measures are attainable. For this to be useful when conducting a search, it is necessary to have detailed knowledge of the environment in which the search is being carried out, so that a basis for a valid comparison will exist.

As described in the body of this paper, the parabolic equation method will often be the method of choice when making propagation loss predictions. However, as we have seen, a knowledge of the environment is again crucial, since other methods are preferable under those circumstances described previously.

In general, good agreement between theory and experimental data, whichever acoustic model is being employed, requires accurate knowledge of such environmental details as the SVP, bottom layering, bottom rigidity (shear), scattering, etc., as well as the range-dependency of these conditions. However, the required environmental information is often unavailable. Furthermore, as acoustic models become more sophisticated they require even more extensive knowledge of the propagation environment. We believe that the primary difficulty today in ocean acoustic modeling is the lack of sufficiently detailed and accurate environmental information for input to the models. To us this appears more crucial than further refinements to the models themselves. As environmental details are generally difficult and expensive to obtain, a knowledge of the accuracy required in the input data to the various models, in order to obtain sufficiently accurate propagation loss results, would seem to be extremely important. An analysis of this crucial question is strongly recommended.

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APPENDIX D

A Notch Filter Employing Coefficients of Equal Magnitude

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Abstract

A nearly ideal notch filter, employing coefficients of equal magnitude, is described. Applications to the design of transmitting antenna arrays are discussed briefly. The construction is based upon earlier work of the author involving polynomials with restricted coefficients.

EDICS category 3.1.5

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Figure Captions

Figure 1. A graph of $|Q(\Theta)|$ for N=60, in dB's.

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Abstract

A nearly ideal notch filter, employing coefficients of equal magnitude, is described. Applications to the design of transmitting antenna arrays are discussed briefly. The construction is based upon earlier work of the author involving polynomials with restricted coefficients.

I. INTRODUCTION

The classical mathematical problem in notch filter design is to produce a polynomial whose magnitude on the unit circle is close to constant in almost all directions, but which has a small number (i.e., 1, 2 or 3) of deep nulls ("notches") at specified points. Such filters are applied, for example, to remove spectral lines from otherwise broadband spectra. In this paper we produce such polynomials having one null, with the added feature that all coefficients have the same magnitude. For convenience this magnitude is assumed to be one. Observe that this "unimodular" property allows the direct application of these polynomials to the design of transmitting antenna arrays which are omnidirectional except for 1 null. This feature is crucial in certain communications areas, where it is desired to null out one listener in a known direction while at the same time, for maximum efficiency, all antenna elements are

broadcasting at full power.

If the polynomial $P(z)$ is of degree $n-1$, it is clear from the Parseval Theorem that its L^2 norm (i.e., RMS value) is exactly $n^{1/2}$, since there are n coefficients each of magnitude 1. Thus, for $|P(z)|$ to be close to constant on $|z|=1$, that constant must be $n^{1/2}$. The question of the existence of such polynomials is a classic one in mathematical analysis. Its study was apparently initiated by Hardy [11,p.199], and furthered by Littlewood [8,9], Erdös [5], Newman [1,2,3,10] and others. A basic result concerning these problems was obtained by the author [4], which paved the way for solutions, by Körner [7] and Kahane [6], of two of the fundamental conjectures in this area. We modify the construction given in [4] to produce nearly ideal filters with one notch.

II. APPROACH AND RESULTS

Our starting point is the polynomial P, of degree N^2-1, given by

$$P(\theta)= \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \exp(2\pi i jkN^{-1}) z^{j+kN}, \quad z=\exp(2\pi i\theta).$$

It is shown in [4] that $P(\theta)$ satisfies:

(i) $|P(mN^{-2})|=N$ for all integers m,

(ii) For any ϵ, $N^{-1}<\epsilon<1/2$, $|P(\theta)|=N+E$ for $\epsilon\leq|\theta|\leq 1/2$, where $|E|<1+2\pi^{-1}+5(\pi\epsilon)^{-1}$,

(iii) For N odd, $P(1/2N)=O(1)$, while for N even, $P((N-1)/(2N^2))=O(1)$, and

(iv) $|P(\theta)|<(2+3\pi^{-1})N + O(1)$ for all θ.

Recent numerical evidence suggests that (iv) can be strengthened to:

(iv') $|P(\theta)|<1.4N$ for all θ.

Thus, as $N\to\infty$, the magnitude of P is asymptotically close to the ideal constant N except for the immediate neighborhood of one point. By a simple change of variables it is clear that this special point can be taken anywhere on the unit circle.

If $P(\theta)$ is changed by removing the first N terms, all of whose coefficients are +1, and then dividing by z^N, there results

$$Q(\theta)= \sum_{k=1}^{N-1} \sum_{j=0}^{N-1} \exp(2\pi i jkN^{-1}) z^{j+(k-1)N}.$$

This Q, which is of degree N^2-N-1, is the desired modification of P. In fact, for $\theta=0$ the terms in the inner sum for Q are identically 0, so that $Q(0)=0$. Also, the N terms removed from P are significant only in the immediate neighborhood of $\theta=0$ (i.e., the special point), so that estimates (ii) and (iv') remain true for Q. Furthermore, it can be shown that the null width of Q is less than 2/N. These properties together with figure 1, which exhibits $|Q(\theta)|$ in dB's as a function of θ for N=60 (Q of degree 3539), show that Q is indeed the nearly ideal notch filter discussed earlier. Once again it is clear that a change of variables allows the relocation of the notch to any desired θ.

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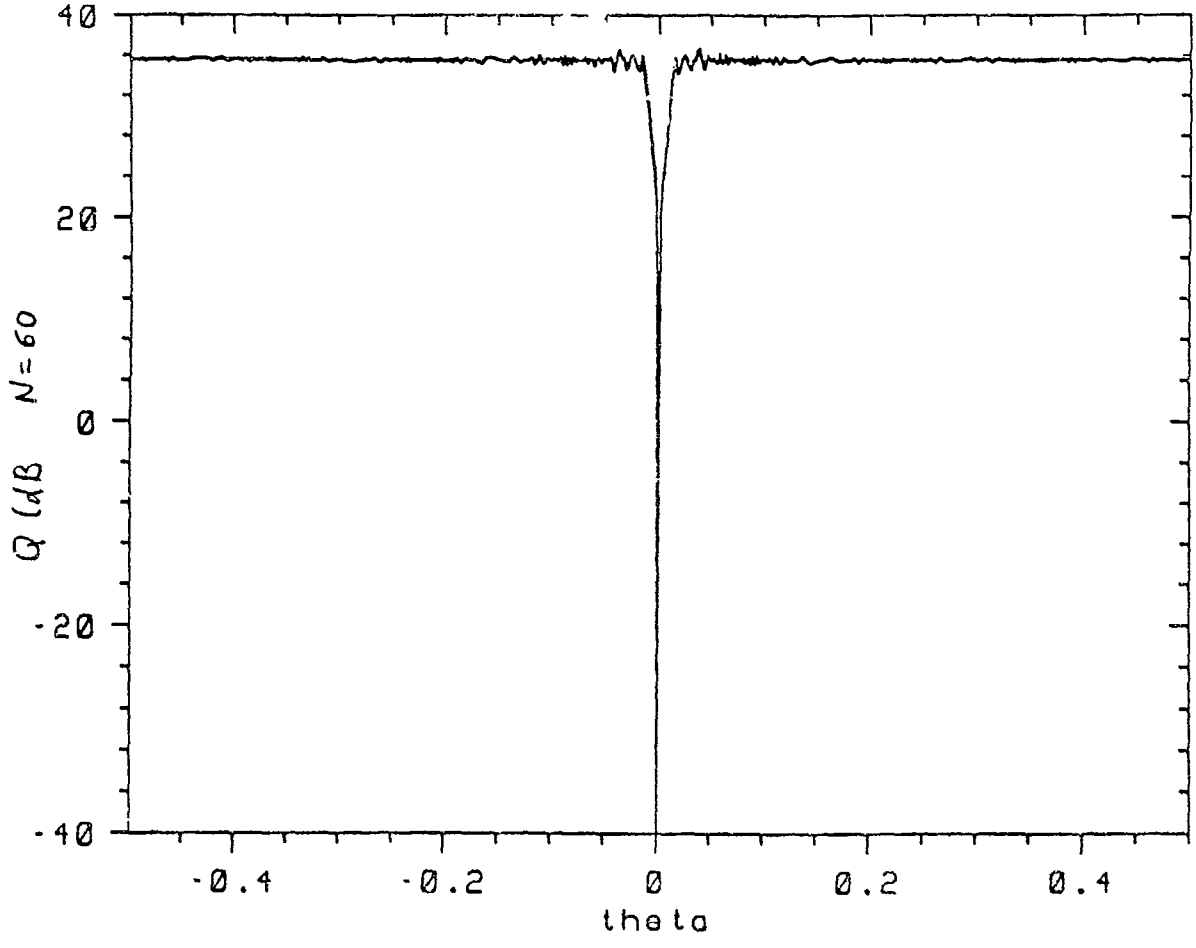

Figure 1. A graph of $|Q(\theta)|$ for $N=60$, in dB's.

APPENDIX E

Null Steering Employing Polynomials with Restricted Coefficients

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ABSTRACT

Several new designs of analytic null steering algorithms for linear arrays are described. Two of them, the δ-Technique and the Positive Coefficient Model, allow for placing an arbitrary number of nulls in arbitrary directions, while maintaining main beam and sidelobe level control. A method of incorporating these deterministic null steering techniques into existing adaptive algorithms, offering the possibility of increases in array performance at small cost, is currently being researched.

Null Steering Employing Polynomials with Restricted Coefficients

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Prometheus Inc.

I. INTRODUCTION

In view of the well-known one-to-one correspondence between polynomials and linear arrays with commensurable separations between elements, as described in detail by Schelkunoff [9], null steering questions involving such arrays translate directly into mathematical problems regarding the locations of zeroes, on the unit circle, of polynomials. Furthermore, physical and electronic limitations placed upon the array elements, such as a maximum allowable transmitting power or a bound on the dynamic range, imply various restrictions upon the coefficients of these polynomials. Here, dynamic range refers to the ratio of the magnitudes of the largest to the smallest weight, or shading coefficient, of the array. Thus, the theoretically challenging question of the placement of zeroes at specified points on the unit circle, of polynomials whose coefficients satisfy certain restrictions, is also a problem of strong practical interest to antenna designers.

The design of filters is another application in which such questions arise. For example, the classical mathematical problem in notch filter design is to produce a polynomial whose magnitude on the unit circle is close to constant in almost all directions, but which has a small number (ie, 1, 2 or 3) of deep nulls ("notches") at specified points. In [5] the construction of [4] is employed to produce such a polynomial having one null, with the added feature that

all coefficients have the same magnitude. Hence, the dynamic range of the notch filters presented in [5] is one.

This paper addresses the null steering application described in the first paragraph. There are several factors which must be considered in the design of null steering algorithms. In addition to the basic problem of placing the nulls the main beam must be steered, the width of the main lobe controlled, and the sidelobe levels must be sufficiently below that of the main lobe. Control of the sidelobe level is usually achieved by attenuating the shading coefficients as one moves away from the center of the array. Often these attenuation factors (Chebyshev, Taylor, etc.) are chosen in advance, and may not be easily altered once the array is in place. This leads directly to a beautiful mathematical question, similar to the peak-factor problem in engineering attacked by Boyd [2], Schroeder [10] and others:

> Given the magnitude of the coefficients of a polynomial P, a finite subset S of the unit circle C, and a point $p \in C$ distinct from those in S, choose the phases of these coefficients so that $P(z)=0$ for all $z \in S$, the maximum on C of $|P(z)|$ occurs at $z=p$, and the maximum of $|P(z)|$ on a subset of C excluding an appropriate interval (the beamwidth) around p is as small as possible.

We consider various subproblems in this paper. Research on the general question is continuing.

II. DIRECT ADAPTIVE NULLING

Currently the most widely used class of null steering methods is known as adaptive nulling [1,3,6,7,8,12]. Adaptive arrays have

developed over the past twenty-five years as the preferred method of reducing the performance deterioration in signal reception systems which is inevitably caused by noise entering the system. These methods, such as the Widrow LMS Algorithm and the Howells-Applebaum sidelobe canceller, are indirect adaptive schemes, they do not explicitly form an estimate of the directions of arrival of interfering sources or explicitly steer nulls in those directions. A scheme in which these two tasks are actually performed can be called a direct adaptive algorithm. Thus one possible approach to the solution of many of the well-known performance problems which arise with existing methods is to complement an appropriate indirect adaptive algorithm with the analytic null steering methods described herein. In this way the actual noise suppression achieved might be enhanced beyond that which would be available through either adaptive or analytic methods exclusively. In addition, since each execution of our analytic algorithms is essentially instantaneous, and since a direct scheme allows much greater use of prior knowledge, such as known jammer locations or known multipaths, this interactive direct method offers the possibility of increases in performance at small cost. Research on this approach, which we call the "Direct Adaptive Nulling System," is in progress.

III. COEFFICIENTS OF EQUAL MAGNITUDE

An important subproblem of the general mathematical question

described earlier is the case when all of the coefficients of the polynomial have the same magnitude, which, by normalization, we can assume to be one. Such phase-only shading occurs, for example, in the design of transmitting arrays which are omnidirectional except for specified nulls. These features are crucial in certain communications areas, where it is desired to null out listeners in known directions while, at the same time, for maximum efficiency, all antenna elements are broadcasting at full power. Also, in order to minimize the relative size of the quantisation steps in a gradient algorithm such as LMS, the coefficient magnitudes should be kept as close as possible to unity [7, p. 153]. Note that this "equimagnitude" property of the coefficients precludes the use of attenuators, with the concomitant savings in electronic hardware.

The most elementary example of the above is the unshaded array - all coefficients are 1. In spite of its simplicity, this uniform array is of practical importance. Observe that in this case the zeroes of the polynomial are almost uniformly spaced around the unit circle, occurring at all of the n+1st (where n is the degree of the polynomial) roots of unity except z=1, where there is a maximum.

At the other extreme is the case where an n-fold zero is required at one point. One application of this, as discussed by Steyskal [11], is to broaden a pattern null so as to null an entire sector. Clearly, by a simple change of variables, this zero-point can be assumed to occur at z=1. It is a straightforward matter to construct such a polynomial with coefficients of magnitude 1; in fact, the coefficients may all be taken to be ± 1. Namely, define P(z) by:

$$P(z)=\prod_{m=0}^{n-1}(1-z^{2^m}) \qquad (1)$$

The problem with this construction is that, although P(z) obviously satisfies the required properties, it does so at very high cost. Since P has degree 2^n-1, its realization requires an array with 2^n elements. We show in the following theorem that, for all but small values of n, this situation may be greatly improved by allowing coefficients to be 0 as well as ± 1. Since this simply means that some array elements are turned off, the dynamic range of the coefficients is not affected in any meaningful way.

Theorem I Let $n \geq 10$. Then there is a polynomial P(z), of degree less than n^3, such that P(z) has an n-fold zero at z=1, and all the coefficients of P are either ± 1 or 0.

Proof of Theorem I Given $n \geq 10$, choose k so that

$$2^k > k^{n^2} \qquad (2).$$

Since $x/\ln(x)$ is increasing for $x>e$, and $10/\ln(10) > 3/\ln(2)$, the choice $k=n^3$ certainly implies (2). Actually, for any $\epsilon>0$, it is clear that for n large enough we may take $k=n^{3-\epsilon}$, but such precision is not necessary here. Now, for each arbitrary subset S of the set of nonnegative integers less than k, let

$$Q(z)=\sum_{m \in S} z^m,$$

and form the vector

$$(Q(1),\ Q'(1),\ Q''(1)/2!,\ldots Q^{(n-1)}(1)/(n-1)!).$$

These are integer vectors, and the largest entry is bounded by

$$\sum_{m=0}^{k-1} m^{n-1} < k^n.$$

Thus there are less than k^{n^2} such vectors. Since there are 2^k subsets S, and so 2^k polynomials Q, equation (2) implies that at least two distinct polynomials, say $Q_1(z)$ and $Q_2(z)$, have the same associated vector. Hence $P(z)=Q_1(z)-Q_2(z)$ is the desired polynomial, and Theorem I is proven.

The idea underlying equation (1), which we call "encapsulation," may also be employed to construct polynomials with coefficients of magnitude 1 that place any number of arbitrary nulls. Namely, we have:

Theorem II For any positive integer n, let $\{z_m\}_{m=1}^{n}$ be an arbitrary set of (not necessarily distinct) points on the unit circle. Then there is a polynomial P(z) with coefficients all of magnitude 1, of degree 2^n-1, satisfying $P(z_m)=0$, $1\leq m\leq n$.

Proof of Theorem II As indicated above, we simply produce an explicit formula for P(z):

$$P(z)=\prod_{m=1}^{n} (z^{2^{m-1}} - z_m^{2^{m-1}}).$$

It is straightforward to see that this $P(z)$ satisfies the required properties, establishing Theorem II.

IV. SIMULTANEOUS NULL STEERING AND MAIN BEAM PLACEMENT

In the previous section we attacked the subproblem, of the general question stated earlier, which arises when the dynamic range of the coefficients is required to be one. We now consider another aspect of the original problem, perhaps of more interest to antenna designers. Namely, how can arbitrary nulls be placed while maintaining a specified main beam direction and specified maximum sidelobe level? We describe two methods, the β-Technique and the Positive Coefficient Model, of achieving these goals.

To set the problem, again let n denote any positive integer, and let $S=\{z_m\}_{m=1}^{n}$ be an arbitrary set of (not necessarily distinct) points on the unit circle. Also, let z_0 be a point on the unit circle distinct from those in S. Our methods allow the placement of zeroes of a p ynomial P at all points in S, while simultaneously having the maximum of $|P|$ on the unit circle occur at $z=z_0$. Furthermore, the difference between $|P(z_0)|$ and the highest sidelobe can be made arbitrarily large. As will be seen, the costs encountered in achieving the last property are an increase in the degree of P, and a loss of control of the dynamic range of the coefficients.

To proceed with the constructions, define the angles $\{\theta_m\}$, $0\le m\le n$, by $z_m=\exp(i\theta_m)$, $-\pi<\theta_m\le\pi$. As before, a simple change of variables allows us to assume $\theta_0=0$, so that $z_0=1$.

Method 1. The β-Technique

Let $\beta=-\cot^{-1}\left[2\sum_{m=1}^{n}\cot(\theta_m/2)\right]$, $z^*=\exp(i\beta)$,

and define $Q(z)$ by

$$Q(z)=(z-z^{*})\prod_{m=1}^{n}(z-z_m).$$

A straightforward calculation shows that $|Q(z)|$ has a relative maximum at $z=1$. Hence, for c a large enough positive integer, $P(z)=(1+z)^{c}Q(z)$ will certainly satisfy the required properties. It can be shown that, in order to guarantee an absolute maximum of $|P(z)|$ at $z=1$, it is sufficient to take $c=[1/\epsilon]$, where $\epsilon=\min|\theta_m|$. Of course, in order to further increase the main lobe level relative to the sidelobes, it will be necessary to take c larger.

Method II. The Positive Coefficient Model

For each m, $1\leq m\leq n$, choose the smallest positive integer k_m such that $\exp(ik_m\theta_m)$ lies in the left half plane, and define $P(z)$ by

$$P(z)=\prod_{m=1}^{n}\left(z^{k_m}-z_m^{k_m}\right)\left(z^{k_m}-\bar{z}^{k_m}\right).$$

Clearly $P(z)$ has the necessary zeroes. Furthermore, all of the coefficients of P are positive, so that the maximum of $|P(z)|$ on the unit circle obviously occurs at $z=1$. Once again it is simple to further increase the main lobe level relative to the sidelobes, by multiplying $P(z)$ by an appropriate positive integer power of $1+z$.

There are two additional points which can be made about the Positive Coefficient Model. One is that its electronic implementation will be greatly simplified as compared to that of arbitrary shading coefficients, since the positivity eliminates the need for phase

shifters. A second is that some control of the dynamic range of the coefficients can be achieved by combining this method with the encapsulation technique discussed earlier, if we again ignore the effects of 0 coefficients.

V. CONCLUSION

Various aspects of a fascinating problem in classical mathematical analysis, with direct applications to antenna array design, have been discussed, and several results obtained. Foremost among these are two analytic methods for placing an arbitrary number of nulls in arbitrary directions, while maintaining main beam and sidelobe level control. A method of incorporating these analytic null steering techniques into existing adaptive algorithms, offering the possibility of increases in array performance at small cost, is currently being researched.

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APPENDIX F

Error Estimates Resulting from the Norms of Certain Noise Covariance Matrices

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Abstract

Precise estimates were previously obtained for the error in the optimum gain of an antenna array arising from errors in the measured noise covariance matrix. It was observed that the error estimates are a function of the character and spatial distribution of the noise field. The purpose of the current work is to analyze this dependence of the previous results upon the quantitative and qualitative aspects of the noise field.

A. Introduction

In [1] the question of the effect of errors in the measured noise covariance matrix upon the gain of an antenna array was addressed. Precise error estimates were obtained for the case of a linear array of n equally spaced elements in a noise field made up of spatially uncorrelated point sources of a single frequency. Note that under these conditions the noise covariance matrix is Hermitian, positive definite and Toeplitz. It was observed in [1] that the error estimates were a function of the character and spatial distribution of the noise field. The purpose of the current work is to analyze this dependence of the previous results upon the quantitative and qualitative aspects of the noise field. Several of the results given below were announced in [2]. A significant improvement in the major result (Theorem C), as well as proofs for the previously announced theorems and a discussion of their importance, is given herein.

B. Results

The notation will be the same as that in [1] and [2], so that n is the number of antenna elements, R is the true noise covariance matrix, $\hat{R}$ is the measured noise covariance matrix, $E=\hat{R}-R$ is the error matrix, $\|\cdot\|$ is a norm on the space of all $n \times n$ matrices, and $\varepsilon=\|E\|$. For weight (i.e. "shading") vector W and signal vector S the gain is given by

$$G=\frac{|(W^{\dagger},S)|^2}{W^{\dagger}RW}.$$

When W is chosen to be an ideal Wiener weight vector,

$$W=R^{-1}S,$$

the resulting G is known as the optimal gain. Since the measured noise covariance matrix $\hat{R}$ is, by necessity, employed in obtaining the actual Weiner weight vector

$$\hat{W}=\hat{R}^{-1}S,$$

the actual gain will differ from the optimal gain G. Denoting this actual gain by $\tilde{G}$, it is seen that

$$\tilde{G}=\frac{(S^{\dagger}\hat{R}^{-1}S)^{2}}{S^{\dagger}\hat{R}^{-1}R\hat{R}^{-1}S}\ .$$

Note that the true noise covariance matrix R appears in the denominator of $\tilde{G}$.

In order to determine the effects of the noise field upon the gain, the following result from [1] will be employed:

Theorem A [1, Corollary II]. For any small enough positive number ξ,

$$\left|\frac{G}{\tilde{G}}-1\right|<16\|R^{-1}\|\cdot\|R\|^{2}\xi^{2}.$$

It is clear from Theorem A that what is now required is an analysis of how the norms of the noise covariance matrix and its inverse depend upon the noise field. As is well known, the determination of the norm of a matrix, except in very special cases, is no easy matter. Even for the particular class of matrices considered here (i.e., Hermitian, positive definite, Toeplitz), precise estimates for the norm are difficult to attain. The most tractable norm to deal with in this

context is defined, for an n x n matrix A, by:

$$\|A\|_2 = \text{Max}\,|y^T A x|,$$

where the maximum is taken over all n-vectors x and y with Euclidean norm 1. If A is Hermitian and positive definite, and if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the (necessarily real and positive) eigenvalues of A written in increasing order, it follows from the standard theory that

$$\|A\|_2 = \lambda_n \text{ and}$$
$$\|A^{-1}\|_2 = (\lambda_1)^{-1}. \qquad (1)$$

Hence, accurate estimates for the smallest and largest eigenvalues of R are required.

Consider a model of an isotropic noise field plus a single point source of noise which is generating a plane wave signal. Let σ_n^2 be the power of the isotropic noise, σ_s^2 the point source power, λ the wavelength of the noise, and assume the array spacing to be $\lambda/2$. In addition, define the ratio Γ by:

$$\Gamma = \frac{\sigma_s^2}{\sigma_s^2 + \sigma_n^2}. \qquad (2)$$

As is well known [LeBlanc, p. 12, for example], the normalized noise covariance matrix under these conditions is given by:

$$R = \begin{bmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \vdots & & & & \\ \rho & \rho & \rho & \cdots & 1 \end{bmatrix} . \qquad (3)$$

A straightforward computation shows that:

$$\text{Eigenvalues } (R) = 1-\rho, 1-\rho, \ldots, 1-\rho, 1+(n-1)\rho,$$
$$\text{so that } \|R\| = 1+(n-1)\rho \text{ and } \|R^{-1}\| = \frac{1}{1-\rho} . \qquad (4)$$

If the model is altered by changing the wavelength of the point source, so that the array spacing remains at one-half wavelength of the isotropic component, but not of the point source, the normalized noise covariance matrix becomes:

$$R = \begin{bmatrix} 1 & \rho c_1 & \rho c_2 & \cdots & \rho c_{n-1} \\ \rho c_{-1} & 1 & \rho c_1 & \cdots & \rho c_{n-2} \\ \rho c_{-2} & \rho c_{-1} & 1 & \cdots & \rho c_{n-3} \\ \vdots & & & & \\ \rho c_{-n+1} & \cdots & & & 1 \end{bmatrix} , \qquad (5)$$

where $c_m = e^{iKd \sin t}$, d=array spacing, t=angle of arrival of plane wave (t=0 is endfire), and K is the wavenumber of the point source. Note that $c_{-m} = \bar{c}_m$ and also that $c_m = c^m$, where $c = e^{iKd \sin t}$.

It is surprising that the eigenvalues of R are identical to the previous case. Namely,

Theorem B. Let R be given by (5). Then the eigenvalues and

norms of R are those appearing in (4).

Proof of Theorem B. Let I denote the identity matrix, U be the matrix all of whose entries are 1, and V be the matrix R of (5) with all of the γ's replaced by 1. Thus $R=(1-\gamma)I+\gamma V$. In addition, V may be factored, giving

$$V=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & c^{n-1} \end{bmatrix} U \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & c^{n-1} \end{bmatrix}.$$

From this factorization it follows that

$$\text{eigenvalues } (\gamma V)=\gamma \text{ eigenvalues } (U)=0,0,\ldots,0,\gamma n.$$

But the above expression for R implies that λ is an eigenvalue of R if and only if $\gamma-1+\lambda$ is an eigenvalue of γV, and Theorem B is proven.

Combining Theorems A and B immediately yields the desired error estimate:

Theorem C. Let R be given by (5), and let G, $\tilde{G}$, ε, γ be defined as above. Then, as long as ε is small enough,

$$\left|\frac{G}{\tilde{G}}-1\right| < 16\,\frac{1}{(1-\gamma)^4}\left(1+(n-1)\gamma\right)^2\varepsilon^2,$$

and this estimate is best possible.

Now consider a model of two point sources of noise, generating plane wave signals of power σ_1^2 and σ_2^2 respectively, plus an isotropic noise field of power σ_n^2. Let K_1 and K_2 be the wave numbers of the point sources, t_1 and t_2 the angles of arrival of the plane waves, d the

array spacing, and assume again that $d=\lambda/2$, where λ is the wavelength of the isotropic noise. Defining T, d_m, e_m, α, and f_m by:

$$T=\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2+\sigma_2^2+\sigma_n^2}\ ,\quad d_m=e^{im K_1 d\cos t_1}\ ,\quad e_m=e^{im K_2 d\cos t_2}\ ,$$

$$\alpha=\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\ ,\quad \text{and}\quad f_m=\alpha d_m+(1-\alpha)e_m\ ,$$

the normalized noise covariance matrix is:

$$R=\begin{bmatrix} 1 & Tf_1 & Tf_2 & \cdots & Tf_{n-1} \\ Tf_{-1} & 1 & Tf_1 & \cdots & Tf_{n-2} \\ Tf_{-2} & Tf_{-1} & 1 & \cdots & Tf_{n-3} \\ \vdots & & & & \vdots \\ Tf_{-n+1} & \cdots & & & 1 \end{bmatrix}\ . \qquad (6)$$

Theorem B can be applied directly to obtain an upper bound for $||R||$. To see this, let A denote the matrix R of (6) with f_m replaced by d_m, and let B be R with f_m replaced by e_m. Thus $R=\alpha A+(1-\alpha)B$, so that

$$||R||\le\alpha||A||+(1-\alpha)||B||=1+(n-1)T. \qquad (7)$$

Clearly estimate (7) cannot be improved without additional assumptions regarding the relative sizes of σ_1^2 and σ_2^2, so that in the general case (7) is best possible.

As often occurs in estimation problems, the lower bound on the eigenvalues of R is considerably more difficult to obtain. It appears necessary to employ asymptotic estimates, such as the following theorem [Grenander & Szego, p.64]:

Theorem D. Let $f(x)$ be a real-valued, 2π-periodic, Lebesgue integrable function with Fourier series $f(x)\sim\sum_{m=-\infty}^{\infty}f_m\exp(imx)$. (Note that $\bar{f}_m=f_{-m}$). For each fixed n, $n=1,2,3,\ldots$, let the (necessarily Hermitian, Toeplitz) matrix (f_{m-k}), $m,k=1,2,\ldots,n$, have (necessarily real)

eigenvalues $\lambda_1^{(n)} \le \lambda_2^{(n)} \le \ldots \le \lambda_n^{(n)}$. Finally, let M_1 be the essential inf of $f(x)$, and let M_2 be its essential sup ($M_1 = -\infty$ and/or $M_2 = \infty$ are allowed). Then

$$M_1 \le \lambda_1^{(n)} \le \lambda_n^{(n)} \le M_2, \text{ and}$$
$$\lim_{n\to\infty} \lambda_1^{(n)} = M_1; \quad \lim_{n\to\infty} \lambda_n^{(n)} = M_2. \tag{8}$$

To apply Theorem D to estimate the smallest eigenvalue of the matrix R given in (6), first approximate R by multiplying each f_m by the "convergence factor" $r^{|m|}$, where $0<r<1$ and r is close to 1, so that now

$$f_m = T r^{|m|} \left(\alpha e^{im K_1 d \cos t_1} + (1-\alpha) e^{im K_2 \cos t_2} \right)$$

and

$$f(x) = 1 + T\alpha \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} r^{|m|} e^{im(K_1 d \cos t_1 + x)} + T(1-\alpha) \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} r^{|m|} e^{im(K_2 d \cos t_2 + x)}$$

$$= 1 + T\alpha \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} r^{|m|} \left(\cos m(K_1 d \cos t_1 + x) + i \sin m(K_1 d \cos t_1 + x) \right) \tag{9}$$
$$+ T(1-\alpha) \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} r^{|m|} \left(\cos m(K_2 d \cos t_2 + x) + i \sin m(K_2 d \cos t_2 + x) \right).$$

Applying the standard formula for the Poisson kernel, along with the facts that the cosine function is even and the sine function is odd, to (9) yields:

$$f(x) = 1 + 2T\alpha \left[\frac{1}{2} \frac{1-r^2}{1 - 2r\cos(K_1 d \cos t_1 + x) + r^2} - \frac{1}{2} \right]$$
$$+ 2T(1-\alpha) \left[\frac{1}{2} \frac{1-r^2}{1 - 2r\cos(K_2 d \cos t_2 + x) + r^2} - \frac{1}{2} \right]. \tag{10}$$

Clearly, for any values of K_1, t_1, K_2 and t_2, there will be values of x such that the first terms in each of the square brackets in (10) can be

arbitrarily close to 0 simultaneously (for r close enough to 1), so that the best possible lower bound for the smallest eigenvalue of R given by (6) is 1-T. Applying (1), (7) and Theorem A to this yields:

Theorem E. Let R be given by (6), and let G, $\tilde{G}$, ε, T be defined as above. Then, as long as ε is small enough,

$$\left| \frac{G}{\tilde{G}} - 1 \right| < 16 \frac{1}{(1-T)^4} \left(1+(n-1)T\right)^2 \varepsilon^2 ,$$

and this estimate is best possible.

Next consider a model where R is a real, tridiagonal matrix. Thus, for some real number b:

$$R = \begin{bmatrix} 1 & b & 0 & \cdots & 0 \\ b & 1 & b & \cdots & 0 \\ 0 & b & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & & 1 \end{bmatrix} . \qquad (11)$$

As is well known [Morgera and Cooper, p. 732 for example], the eigenvalues of this R are:

$$\lambda_m = 1 - 2b \cos \frac{\pi m}{n+1}, \quad m = 1, 2, \cdots, n . \qquad (12)$$

Assuming $2b<1$, in order to guarantee that R is positive definite, it is clear that the best possible bounds for the eigenvalues given in (12) are:

$$1-2b<\lambda_1<\lambda_n<1+2b. \qquad (13)$$

Combining (1) and (13) with Theorem A proves:

Theorem F. Let R be given by (11), assume $2b<1$, and let G, $\tilde{G}$, ε, be defined as above. Then, as long as ε is small enough,

$$\left| \frac{G}{\tilde{G}} - 1 \right| < 16 \frac{1}{(1-2b)^4} (1+2b)^2 \varepsilon^2,$$

and this estimate is best possible.

Finally, the complete eigenspectrum for real, symmetric matrices of orders 3 and 4, and the partial eigenspectrum for such matrices of order 5, have been computed explicitly. By various complex algebraic manipulations, with

$$R_3 = \begin{bmatrix} 1 & a & b \\ a & 1 & a \\ b & a & 1 \end{bmatrix}, \quad R_4 = \begin{bmatrix} 1 & a & b & c \\ a & 1 & a & b \\ b & a & 1 & a \\ c & b & a & 1 \end{bmatrix}, \text{ and } \quad R_5 = \begin{bmatrix} 1 & a & b & c & d \\ a & 1 & a & b & c \\ b & a & 1 & a & b \\ c & b & a & 1 & a \\ d & c & b & a & 1 \end{bmatrix},$$

the following table may be derived:

n	Eigenvalues of R_n	Eigenvectors of R_n
3	$1-b,\ 1+\frac{b\pm\sqrt{b^2+8a^2}}{2}$	$\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\ \gamma \\ 1\end{bmatrix}$, where $\gamma = \frac{4a}{b\pm\sqrt{b^2+8a^2}}$
4	$1+\frac{a+c\pm u}{2},\ 1-\frac{a+c\mp v}{2}$	$\begin{bmatrix}1\\ \frac{a-c\pm u}{2} \\ \frac{a-c\pm u}{2} \\ 1\end{bmatrix}, \begin{bmatrix}1\\ \frac{-a+c\pm v}{2(a-b)} \\ \frac{a-c\mp v}{2(a-b)} \\ 1\end{bmatrix}$, where $u=\sqrt{(a-c)^2+4(a+b)^2}$, $v=\sqrt{(a-c)^2+4(a-b)^2}$
5	$(a-c)\beta+1-b$	$[\beta\ 1\ 0\ -1\ -\beta]^T$, where $\beta = \frac{b-d\pm\sqrt{(b-d)^2+4(a-c)^2}}{2(a-c)}$

Table 1. Eigenvalues and Eigenvectors of Real, Symmetric Matrices

Concentrating for the moment on n=3, assume that there is a real-valued point source of noise of the form $\sigma_s^2 \cos(Kd \cos t)$ in an isotropic noise field of power σ_n^2. Defining $\mathcal{J}$ by (2) and setting $w=Kd \cos t$, the normalized noise covariance matrix is now:

$$R_3 = \begin{bmatrix} 1 & \mathcal{J}\cos w & \mathcal{J}\cos 2w \\ \mathcal{J}\cos w & 1 & \mathcal{J}\cos w \\ \mathcal{J}\cos 2w & \mathcal{J}\cos w & 1 \end{bmatrix} . \qquad (14)$$

The eigenvalues of R_3 may be computed exactly, and the result is that they are $1-\mathcal{J}$, $1-\mathcal{J}\cos 2w$, and $1+\mathcal{J}+\mathcal{J}\cos 2w$. Applying (1) and Theorem A yields:

Theorem G. Let R be given by (14), and let G, $\tilde{G}$, ε, $\mathcal{J}$ be defined as above. Then, as long as ε is small enough,

$$\left| \frac{G}{\tilde{G}} - 1 \right| < 16 \frac{1}{(1-\mathcal{J})^4} (1+2\mathcal{J})^2 \varepsilon^2 ,$$

and this estimate is best possible to within a factor of 4.
A similar result holds for n=4. The complete eigenspectrum for n=5 was not derived in a meaningful form.

C. Conclusion

It is clear from the cases examined in Theorems C through G that the angle of arrival of a plane wave from a point source of noise is not the crucial issue, when considering the accuracy required in measuring the noise covariance matrix of a linear array of equally spaced elements. The overiding concern is the relationship between the power of the point source and the power of the isotropic background noise. If

the ratio of the point source power to the isotropic power is very large, then extremely accurate measurements of the noise covariance matrix are required in order to get a reasonable estimate of the gain of the array. The same applies when there are two localized noise sources in an isotropic background, such as occurs with torpedo flow noise and structural vibrations. Note the concurrence of these conclusions with those of Hudson, who observes that the noise covariance matrix will usually be ill-conditioned when there is a relatively weak uncorrelated noise component in interference that is due to a small number of discrete sources (less than the number of array elements) [4, p.136].

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